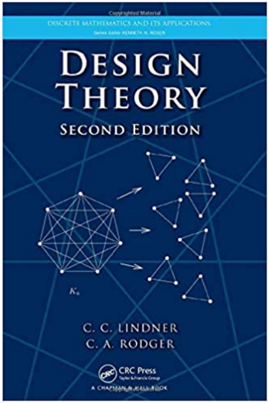
Design Theory

Chapter 2. λ-Fold Triple Systems

2.3. 2-Fold Triple Systems—Proofs of Theorems



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Proof. We use Exercise 2.3.5 which states that if T contains at least $v(v-1)/3$ triples and if every pair of elements of S belongs to at least two triples of T, then (S, T) is a 2-fold triple system of order v. In Note 2.3.A, we see in the case $v = 3n$ that there are $2n$ Type 1 triples and $6\binom{n}{2} = \dfrac{6n(n-1)}{2} = 3n(n-1)$ Type 2 triples. The total number of triples in this case is then $2n + 3n(n-1) = 3n^2 - n = n(3n-1)$ $= (3n)(3n-1)/3 = v(v-1)/3$.

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Theorem 2.3.7 (continued 1)

Proof (continued). Consider the case $v = 3n$. Every pair of the form (i, j_1), (i, j_2), where $i \in Q$ and $j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2$, appears twice in the Type 1 triples. Every pair of the form (i_1, j), (i_2, j), where $i_1 \neq i_2$, $i_1, i_2 \in Q$, and $j \in \{1, 2, 3\}$, appears twice in the Type 2 triples (in the first two positions of the triples as given in Note 2.3.A).

Theorem 2.3.7 (continued 1)

Proof (continued). Consider the case $v = 3n$. Every pair of the form (i, j_1), (i, j_2), where $i \in Q$ and $j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2$, appears twice in the Type 1 triples. Every pair of the form (i_1, j), (i_2, j), where $i_1 \neq i_2$, $i_1, i_2 \in Q$, and $j \in \{1, 2, 3\}$, appears twice in the Type 2 triples (in the first two positions of the triples as given in Note 2.3.A). Every pair of the form $(i_1, 1)$, $(i_2, 2)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the first two Type 2 triples as given in Note 2.3.A); since $(Q, \circ)$ is idempotent then in the Type 2 triples $x \circ y, y \circ x \notin \{x, y\}$ so that there is not another repetition of the pairs (i, j_1), (i, j_2) where $j_1 = 1$ and $j_2 = 2$ (this is where the idempotent property is needed).

Theorem 2.3.7 (continued 1)

Proof (continued). Consider the case $v = 3n$. Every pair of the form (i, j_1), (i, j_2), where $i \in Q$ and $j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2$, appears twice in the Type 1 triples. Every pair of the form (i_1, j), (i_2, j), where $i_1 \neq i_2$, $i_1, i_2 \in Q$, and $j \in \{1, 2, 3\}$, appears twice in the Type 2 triples (in the first two positions of the triples as given in Note 2.3.A). Every pair of the form $(i_1, 1)$, $(i_2, 2)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the first two Type 2 triples as given in Note 2.3.A); since $(Q, \circ)$ is idempotent then in the Type 2 triples $x \circ y, y \circ x \notin \{x, y\}$ so that there is not another repetition of the pairs (i, j_1), (i, j_2) where $j_1 = 1$ and $j_2 = 2$ (this is where the idempotent property is needed). Similarly, every pair of the form $(i_1, 2)$, $(i_2, 3)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the second two Type 2 triples as given in Note 2.3.A). Also, every pair of the form $(i_1, 3)$, $(i_2, 1)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the third two Type 2 triples as given in Note 2.3.A). This includes all possible types of pairs, so that a 2-fold triple system of order $v \equiv 0 \pmod 3$ exists.

Theorem 2.3.7 (continued 1)

Proof (continued). Consider the case $v = 3n$. Every pair of the form (i, j_1), (i, j_2), where $i \in Q$ and $j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2$, appears twice in the Type 1 triples. Every pair of the form (i_1, j), (i_2, j), where $i_1 \neq i_2$, $i_1, i_2 \in Q$, and $j \in \{1, 2, 3\}$, appears twice in the Type 2 triples (in the first two positions of the triples as given in Note 2.3.A). Every pair of the form $(i_1, 1)$, $(i_2, 2)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the first two Type 2 triples as given in Note 2.3.A); since $(Q, \circ)$ is idempotent then in the Type 2 triples $x \circ y, y \circ x \notin \{x, y\}$ so that there is not another repetition of the pairs (i, j_1), (i, j_2) where $j_1 = 1$ and $j_2 = 2$ (this is where the idempotent property is needed). Similarly, every pair of the form $(i_1, 2)$, $(i_2, 3)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the second two Type 2 triples as given in Note 2.3.A). Also, every pair of the form $(i_1, 3)$, $(i_2, 1)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the third two Type 2 triples as given in Note 2.3.A). This includes all possible types of pairs, so that a 2-fold triple system of order $v \equiv 0 \pmod 3$ exists.

Theorem 2.3.7 (continued 2)

Theorem 2.3.7. The spectrum for 2-fold triple systems is precisely the of all $v \equiv 0$ or 1 (mod 3).

Proof (continued). Consider the case $v = 3n + 1$. Every pair of the form (i, j_1), (i, j_2), where $i \in Q$ and $j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2$, appears twice in the Type 1 triples. Every pair of the form ∞, (i, j), where $i \in Q$ and $j \in \{1, 2, 3\}$, appears twice in the Type 1 triples. As in the case $v = 3n$, the Type 2 triples cover the remaining types of pairs exactly twice (namely, (i_1, j_1), (i_2, j_2) where $i_1, i_2 \in Q$, $i_1 \neq i_2$, $j_1, j_2 \in \{1, 2, 3\}$). Since this includes all possible types of pairs, so that a 2-fold triple system of order $v \equiv 1 \pmod 3$ exists.

Therefore, a 2-fold triple system of order v exists if and only if $v \equiv 0$ or 1 (mod 3). □

Theorem 2.3.7 (continued 2)

Theorem 2.3.7. The spectrum for 2-fold triple systems is precisely the of all $v \equiv 0$ or $1 \pmod 3$.

Proof (continued). Consider the case $v = 3n + 1$. Every pair of the form (i, j_1), (i, j_2), where $i \in Q$ and $j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2$, appears twice in the Type 1 triples. Every pair of the form ∞, (i, j), where $i \in Q$ and $j \in \{1, 2, 3\}$, appears twice in the Type 1 triples. As in the case $v = 3n$, the Type 2 triples cover the remaining types of pairs exactly twice (namely, (i_1, j_1), (i_2, j_2) where $i_1, i_2 \in Q$, $i_1 \neq i_2$, $j_1, j_2 \in \{1, 2, 3\}$). Since this includes all possible types of pairs, so that a 2-fold triple system of order $v \equiv 1 \pmod 3$ exists.

Therefore, a 2-fold triple system of order v exists if and only if $v \equiv 0$ or $1 \pmod 3$. □