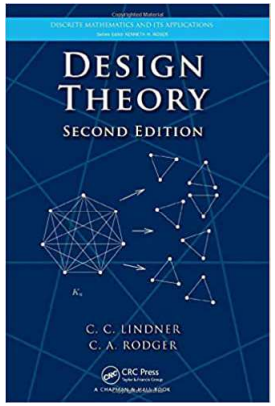
Design Theory

Chapter 3. Quasigroup Identities and Graph Decompositions

3.1. Quasigroup Identities—Proofs of Theorems



Lemma 3.1.A

Lemma 3.1.A. Let R be an $n^2 \times 3$ with the property that each ordered pair (a, b) satisfies the following: (1) a in the first column and b in the second column in exactly once (say when c is in the third column), (2) a in the first column and b in the third column exactly once (say when x is in the second column), and (3) a in the second column and b in the third column in exactly once (say when y is in row one). Define $a \circ b = c$ based on property (1). Then $(Q, \circ)$ is a quasigroup.

Proof. Let $a, b \in Q$. We have $a \circ b = c$ if and only if $((a, b), c) := (a, b, c) \in R$. For every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions (namely, x and y where these are determined from the unique triple of type (2) of the form (a, x, b)) and the unique triple of type (3) of the form (y, a, b) in R respectively). Therefore, by definition, $(Q, \circ)$ is a quasigroup. □

Lemma 3.1.B

Lemma 3.1.B. If R is an $n^2 \times 3$ orthogonal array, the for every $\alpha \in S_3$ we have $R\alpha$ is also an orthogonal array.

Proof. Let $(Q, \circ)$ be the quasigroup equivalent to orthogonal array R. Then $a \circ b = c$ if and only if (a, b, c) is a row of R (by the definition of "orthogonal array"). We need to show that $R\alpha$ is equivalent to some quasigroup. Notice that the entries of R and $R\alpha$, and the elements of Q are all the same.

Define binary operation $\circ'$ on the elements Q as $a \circ' b = c$ if and only if (a, b, c) is a row of $R\alpha$ (since α is a bijection on the rows of R, and the rows of R are not repeated, then the rows of $R\alpha$ are not repeated). By the definition of quasigroup, we need to show that for every pair of entries in $R\alpha$, a, b, the equations $a \circ' x = b$ and $y \circ' a = b$ have unique solutions. Now in $(Q, \circ)$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions since $(Q, \circ)$ is a quasigroup. So we relate the equations in $\circ'$ to the equations in $\circ$ using permutations in S_3.

Lemma 3.1.B (continued 1)

Proof (continued). Consider what happens when we apply $\alpha \in S_3$ to the position of the variables (first, second, or third position) in the equations $a \circ' x = b$ and $y \circ' a = b$:

Permutation	Permuted Equations
$(1)(2)(3)$	$a \circ' x = b$, $y \circ' a = b$
$(1, 2)$	$x \circ' a = b$, $a \circ' y = b$
$(1, 3)$	$b \circ' x = a$, $b \circ' a = y$
$(2, 3)$	$a \circ' b = x$, $y \circ' b = a$
$(1, 2, 3)$	$b \circ' a = x$, $b \circ' y = a$
$(1, 3, 2)$	$x \circ' b = a$, $a \circ' b = y$

Taking the "variable" always as x and the "constants" as a and b, then each of the permuted equations is of one of the forms (we take constant a to the left of b in these forms): $x \circ' a = b$, $a \circ' x = b$, or $a \circ' b = x$. Since $\circ'$ is a binary operation, every equation of the form $a \circ' b = x$ has a unique solution x.

Lemma 3.1.B (continued 2)

Lemma 3.1.B. If R is an $n^2 \times 3$ orthogonal array, the for every $\alpha \in S_3$ we have $R\alpha$ is also an orthogonal array.

Proof (continued). An equation of the form $x \circ a = b$ has a unique solution in $(Q, \circ)$, so that (x, a, b) is a row of R exactly once. Since α is a bijection of the rows of R, then (x, a, b) is a row of $R\alpha$ exactly once. Therefore, $x \circ' a = b$ has a unique solution. Similarly, an equation of the form $a \circ x = b$ has a unique solution in $(Q, \circ)$ so that (a, x, b) is a row of R exactly once, α is a bijection of the rows of R and (x, a, b) is a row of $R\alpha$ exactly once, therefore $a \circ' ax = b$ has a unique solution. That is, $(Q, \circ')$ is a quasigroup. Finally, $R\alpha$ is the orthogonal array equivalent to this quasigroup, as claimed. □