Design Theory

Chapter 8. Intersections of Steiner Triple Systems

8.1. Teirlinck's Algorithm—Proofs of Theorems

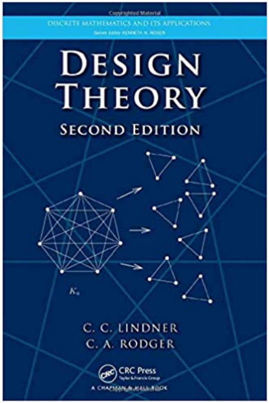

Table of contents

Theorem 8.1.A. The Reduction Algorithm

Theorem 8.1.A. The Reduction Algorithm. Let (S, T_1) and (S, T_2) be any two STS(n)s and suppose that $\{1, 2, 3\} \in T_1 \cap T_2$ and $|S(3)| < n$. Then there exists a transposition α such that $T_1 \cap T_2\alpha \subseteq T_1 \cap T_2$ and $|T_1 \cap T_2\alpha| < |T_1 \cap T_2|$.

Proof. Let x be any element of S that does not belong to $S(3)$ (such an element exists since $S(3) < n$) and let $\alpha = (3, x)$. There are three kinds of triples of elements of set S: those containing neither x nor 3, those containing both x and 3, and those containing exactly one of x and 3. First $T_1 \cap T_2\alpha$ and $T_1 \cap T_2$ contain exactly the same triples that contain neither x nor 3 (since these triples are fixed under α). Similarly, $T_1 \cap T_2\alpha$ contains the triple $\{3, x, a\}$ if $\{3, x, a\} \in T_1 \cap T_2$ (since this triple is also fixed under α).

Theorem 8.1.A. The Reduction Algorithm

Theorem 8.1.A. The Reduction Algorithm. Let (S, T_1) and (S, T_2) be any two STS(n)s and suppose that $\{1, 2, 3\} \in T_1 \cap T_2$ and $|S(3)| < n$. Then there exists a transposition α such that $T_1 \cap T_2\alpha \subseteq T_1 \cap T_2$ and $|T_1 \cap T_2\alpha| < |T_1 \cap T_2|$.

Proof. Let x be any element of S that does not belong to $S(3)$ (such an element exists since $S(3) < n$) and let $\alpha = (3, x)$. There are three kinds of triples of elements of set S: those containing neither x nor 3, those containing both x and 3, and those containing exactly one of x and 3. First $T_1 \cap T_2\alpha$ and $T_1 \cap T_2$ contain exactly the same triples that contain neither x nor 3 (since these triples are fixed under α). Similarly, $T_1 \cap T_2\alpha$ contains the triple $\{3, x, a\}$ if $\{3, x, a\} \in T_1 \cap T_2$ (since this triple is also fixed under α). Thus $(T_1 \cap T_2) \setminus I \subseteq T_1 \cap T_2\alpha$, where I is the set of all triples in $T_1 \cap T_2$ containing exactly one of x and 3. Hence $T_1 \cap T_2\alpha = ((T_1 \cap T_2) \setminus I) \cup P$, where P is some set of triples containing exactly one of x and 3. ASSUME $P \neq \varnothing$.

Theorem 8.1.A. The Reduction Algorithm

Theorem 8.1.A. The Reduction Algorithm. Let (S, T_1) and (S, T_2) be any two STS(n)s and suppose that $\{1,2,3\} \in T_1 \cap T_2$ and $|S(3)| < n$. Then there exists a transposition α such that $T_1 \cap T_2\alpha \subseteq T_1 \cap T_2$ and $|T_1 \cap T_2\alpha| < |T_1 \cap T_2|$.

Proof. Let x be any element of S that does not belong to $S(3)$ (such an element exists since $S(3) < n$) and let $\alpha = (3, x)$. There are three kinds of triples of elements of set S: those containing neither x nor 3, those containing both x and 3, and those containing exactly one of x and 3. First $T_1 \cap T_2\alpha$ and $T_1 \cap T_2$ contain exactly the same triples that contain neither x nor 3 (since these triples are fixed under α). Similarly, $T_1 \cap T_2\alpha$ contains the triple $\{3, x, a\}$ if $\{3, x, a\} \in T_1 \cap T_2$ (since this triple is also fixed under α). Thus $(T_1 \cap T_2) \setminus I \subseteq T_1 \cap T_2\alpha$, where I is the set of all triples in $T_1 \cap T_2$ containing exactly one of x and 3. Hence $T_1 \cap T_2\alpha = ((T_1 \cap T_2) \setminus I) \cup P$, where P is some set of triples containing exactly one of x and 3. ASSUME $P \neq \varnothing$.

Theorem 8.1.A. The Reduction Algorithm (continued 1)

Proof (continued). ASSUME $P \neq \varnothing$. Then P either contains a triple containing x and not 3, or a triple containing 3 and not x. We consider these two cases.

(i) Consider the case where $\{x, a, b\} \in P$. Then $\{x, a, b\} \in T_1 \cap T_2\alpha$ and so $\{3, a, b\} \in T_2$ (since $\{3, a, b\}\alpha = \{3, a, b\}(3, x) = \{x, a, b\}$). Also $\{x, a, b\} \in T_1$. But $\{x, a, b\} \in T_1$ and $\{3, a, b\} \in T_2 \setminus \{1, 2, 3\}$ means that $x \in S(3)$ (by the definition of $A(3)$). But x was chosen such that $x \notin S(3)$, a CONTRADICTION. So the assumption that $\{x, a, b\} \in P$ is false and $\{x, a, b\} \notin P$.

Theorem 8.1.A. The Reduction Algorithm (continued 1)

Proof (continued). ASSUME $P \neq \varnothing$. Then P either contains a triple containing x and not 3, or a triple containing 3 and not x. We consider these two cases.

(i) Consider the case where $\{x, a, b\} \in P$. Then $\{x, a, b\} \in T_1 \cap T_2\alpha$ and so $\{3, a, b\} \in T_2$ (since $\{3, a, b\}\alpha = \{3, a, b\}(3, x) = \{x, a, b\}$). Also $\{x, a, b\} \in T_1$. But $\{x, a, b\} \in T_1$ and $\{3, a, b\} \in T_2 \setminus \{1, 2, 3\}$ means that $x \in S(3)$ (by the definition of $A(3)$). But x was chosen such that $x \notin S(3)$, a CONTRADICTION. So the assumption that $\{x, a, b\} \in P$ is false and $\{x, a, b\} \notin P$.

(ii) Consider the case where $\{3, a, b\} \in P$. Then $\{3, a, b\} \in T_1 \cap T_2\alpha$ and so $\{x, a, b\} \in T_2$ (since $\{x, a, b\}\alpha = \{x, a, b\}(3, x) = \{3, a, b\}$). Also $\{3, a, b\} \in T_1$. But $\{3, a, b\} \in T_1$ and $\{x, a, b\} \in T_2 \setminus \{1, 2, 3\}$ means that $x \in S(3)$ (by the definition of $B(3)$). But x was chosen such that $x \notin S(3)$, a CONTRADICTION. So the assumption that $\{3, a, b\} \in P$ is false and $\{3, a, b\} \notin P$.

Theorem 8.1.A. The Reduction Algorithm (continued 1)

Proof (continued). ASSUME $P \neq \varnothing$. Then P either contains a triple containing x and not 3, or a triple containing 3 and not x. We consider these two cases.

(i) Consider the case where $\{x, a, b\} \in P$. Then $\{x, a, b\} \in T_1 \cap T_2\alpha$ and so $\{3, a, b\} \in T_2$ (since $\{3, a, b\}\alpha = \{3, a, b\}(3, x) = \{x, a, b\}$). Also $\{x, a, b\} \in T_1$. But $\{x, a, b\} \in T_1$ and $\{3, a, b\} \in T_2 \setminus \{1, 2, 3\}$ means that $x \in S(3)$ (by the definition of $A(3)$). But x was chosen such that $x \notin S(3)$, a CONTRADICTION. So the assumption that $\{x, a, b\} \in P$ is false and $\{x, a, b\} \notin P$.

(ii) Consider the case where $\{3, a, b\} \in P$. Then $\{3, a, b\} \in T_1 \cap T_2\alpha$ and so $\{x, a, b\} \in T_2$ (since $\{x, a, b\}\alpha = \{x, a, b\}(3, x) = \{3, a, b\}$). Also $\{3, a, b\} \in T_1$. But $\{3, a, b\} \in T_1$ and $\{x, a, b\} \in T_2 \setminus \{1, 2, 3\}$ means that $x \in S(3)$ (by the definition of $B(3)$). But x was chosen such that $x \notin S(3)$, a CONTRADICTION. So the assumption that $\{3, a, b\} \in P$ is false and $\{3, a, b\} \notin P$.

Theorem 8.1.A. The Reduction Algorithm (continued 2)

Theorem 8.1.A. The Reduction Algorithm. Let (S, T_1) and (S, T_2) be any two STS(n)s and suppose that $\{1,2,3\} \in T_1 \cap T_2$ and $|S(3)| < n$. Then there exists a transposition α such that $T_1 \cap T_2\alpha \subseteq T_1 \cap T_2$ and $|T_1 \cap T_2\alpha| < |T_1 \cap T_2|$.

Proof (continued). Therefore $P = \varnothing$ and $T_1 \cap T_2\alpha = ((T_1 \cap T_2) \setminus I) \cup P = (T_1 \cap T_2) \setminus I$. Since $\{1,2,3\} \in T_1 \cap T_2$ (by hypothesis) then $\{1,2,3\} \in I$ and $|I| \geq 1$. Hence, $|T_1 \cap T_2\alpha| < |T_1 \cap T_2|$, as claimed. □

Theorem 8.1.b. Teirlinck's Algorithm

Theorem 8.1.B. Teirlinck's Algorithm. Let (S, T_1) and (S, T_2) be any two STS(n)s and suppose that $\{1, 2, 3\} \in T_1 \cap T_2$ and $S(3) = S$. Then there exists a transposition α such that $T_1 \cap T_2\alpha$ contains a triple t and an element $e \in t$ such that $|S(e)| < n$ (where this spread is with respect to triple t) and $|T_1 \cap T_1\alpha| \leq |T_1 \cap T_2|$.

Proof. We number the steps in the proof to match up with image of Teirlink's Algorithm on page 173 of the textbook. (1) We have $\{1, 2, 3\} \in T_1 \cap T_2$ by hypothesis. (2) Let $\{3, x, y\}$ be a triple in T_2 other than the triple $\{1, 2, 3\}$ (so that neither x nor y is 1 or 2). (3) There is a unique triple in T_1 which contains both x and y (by the definition of Steiner triple system), say $\{x, y, c\}$. Then $c \in A(3)$. Since $S(3) = S$ then $\{1, 2, 3\}$, $A(3)$, and $B(3)$ are pairwise disjoint by Note 8.1.A, so that $c \notin \{1, 2, 3\}$. (4) There is a unique triple in T_2 which contains both 3 and c, say $\{3, c, d\}$.

Theorem 8.1.b. Teirlinck's Algorithm

Theorem 8.1.B. Teirlinck's Algorithm. Let (S, T_1) and (S, T_2) be any two STS(n)s and suppose that $\{1, 2, 3\} \in T_1 \cap T_2$ and $S(3) = S$. Then there exists a transposition α such that $T_1 \cap T_2\alpha$ contains a triple t and an element $e \in t$ such that $|S(e)| < n$ (where this spread is with respect to triple t) and $|T_1 \cap T_1\alpha| \leq |T_1 \cap T_2|$.

Proof. We number the steps in the proof to match up with image of Teirlink's Algorithm on page 173 of the textbook. (1) We have $\{1, 2, 3\} \in T_1 \cap T_2$ by hypothesis. (2) Let $\{3, x, y\}$ be a triple in T_2 other than the triple $\{1, 2, 3\}$ (so that neither x nor y is 1 or 2). (3) There is a unique triple in T_1 which contains both x and y (by the definition of Steiner triple system), say $\{x, y, c\}$. Then $c \in A(3)$. Since $S(3) = S$ then $\{1, 2, 3\}$, $A(3)$, and $B(3)$ are pairwise disjoint by Note 8.1.A, so that $c \notin \{1, 2, 3\}$. (4) There is a unique triple in T_2 which contains both 3 and c, say $\{3, c, d\}$.

Theorem 8.1.b. Teirlinck's Algorithm (continued)

Proof. (5) There is a unique triple in T_1 which contains both c and d, say $\{c, d, e\}$. Then $e \in A(3)$. Again, since $S(3) = S$ then $\{1, 2, 3\}$, $A(3)$, and $B(3)$ are pairwise disjoint by Note 8.1.A, so that $e \notin \{1, 2, 3\}$. (6) Let α be the transposition $(3, e)$.

We now consider the two STSs (S, T_1) and $(S, T_2\alpha)$. Set of triples $T_2\alpha$ contains $\{3, c, d\}\alpha = \{c, d, e\}$ and so $\{c, d, e\} \in T_1 \cap T_2\alpha$; set $t = \{c, d, e\}$. Now $\{3, x, y\} \in T_2$ by (2) so $\{e, x, y\} \in T_2\alpha$, and $\{c, x, y\} \in T_1$ by (3). So with respect to $t = \{c, d, e\}$ we have $c \in A(e)$. But then $c \in \{c, d, e\}$ and $c \in A(e)$, so by Note 8.1.A we have that (with respect to t) $S(e) \neq S$ and hence $S(e) < n$, as claimed. It is to be shown in Exercise 8.1.14 that $|T_1 \cap T_2\alpha| \leq |T_1 \cap T_2|$, as claimed. □

Theorem 8.1.b. Teirlinck's Algorithm (continued)

Proof. (5) There is a unique triple in T_1 which contains both c and d, say $\{c, d, e\}$. Then $e \in A(3)$. Again, since $S(3) = S$ then $\{1, 2, 3\}$, $A(3)$, and $B(3)$ are pairwise disjoint by Note 8.1.A, so that $e \notin \{1, 2, 3\}$. (6) Let α be the transposition $(3, e)$.

We now consider the two STSs (S, T_1) and $(S, T_2\alpha)$. Set of triples $T_2\alpha$ contains $\{3, c, d\}\alpha = \{c, d, e\}$ and so $\{c, d, e\} \in T_1 \cap T_2\alpha$; set $t = \{c, d, e\}$. Now $\{3, x, y\} \in T_2$ by (2) so $\{e, x, y\} \in T_2\alpha$, and $\{c, x, y\} \in T_1$ by (3). So with respect to $t = \{c, d, e\}$ we have $c \in A(e)$. But then $c \in \{c, d, e\}$ and $c \in A(e)$, so by Note 8.1.A we have that (with respect to t) $S(e) \neq S$ and hence $S(e) < n$, as claimed. It is to be shown in Exercise 8.1.14 that $|T_1 \cap T_2\alpha| \leq |T_1 \cap T_2|$, as claimed. □