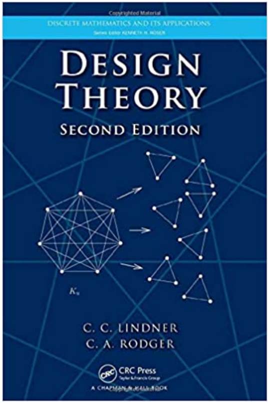
Design Theory

Supplement. Packings and Coverings for Mendelsohn and Directed Triple Systems



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Theorem DPC.A

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1. if $v \equiv 0$ or $1 \pmod 3$ then $L = \varnothing$, and
2. if $v \equiv 2 \pmod 3$ then $L = D_2$.

Proof for $v \equiv 8 \pmod{12}$. First, for $v \equiv 2 \pmod 3$ we have that the arc set $A(D_v)$ satisfies $|A(D_v)| = v(v-1) \equiv 2 \pmod 3$, so a packing with a leave L consisting of two arcs would be maximal.

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Case 3. Suppose $v \equiv 8 \pmod{12}$, say $v = 12t + 8$. Let $S = \{0, 1, 2, \ldots, v-3, x, y\} = \{0, 1, 2, \ldots, 12t+2, x, y\}$. Consider the collection of directed/transitive triples T:

$$\{(0, x, 5t+4)_D, (0, y, 7t+5)_D\} \cup \{(0, 3t+2-i, 3t+3+i)_D \mid i = 0, 1, \ldots, t\}$$

$$\cup \{(0, 5t+3-i, 5t+5+i)_D, (0, 7t+6+i, 7t+4-i)_D \mid i = 0, 1, \ldots, t-1\}$$

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$$\cup \{(0, 9t+6+i, 9t+5-i)_D \mid i = 0, 1, \ldots, t-1\}.$$

These triples, along with their images under the powers of the permutation $(x)(y)(0, 1, \ldots, 12t+5)$, form a packing of D_v with the leave $L = D_2$ where $A(L) = \{(x, y), (y, x)\}$. □

Theorem DPC.D

Theorem DPC.D. A minimum covering of D_v with Mendelsohn triples satisfies:

1. if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$, then $P = \varnothing$,
2. if $v = 6$ then $P = C_3$, and
3. if $v \equiv 2 \pmod 3$ then P has four arcs and may be two disjoint copies of D_2, any orientation of a 4-cycle, or two copies of D_2 which share a single vertex.

Proof for $v \equiv 2$ (mod 6). First, for $v \equiv 2 \pmod 3$ we have that the arc set $A(D_v)$ satisfies $|A(D_v)| = v(v-1) \equiv 2 \pmod 3$. The total degree (i.e., the in-degree plus the out-degree) of each vertex of D_v is $2(v-1)$ and the total degree of each vertex of a Mendelsohn triple is 2. So any covering of D_v with Mendelsohn triples will have a padding P with each vertex of even total-degree. So a covering of D_v cannot have a padding that contains a single arc. Hence a covering with $|A(P)| = 4$ would be minimal.

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Theorem DPC.D (continued 1)

Proof for $v \equiv 2 \pmod 6$, continued. Case 1a. Suppose $v = 8$ and P is two disjoint copies of D_2. Let $S = \{0, 1, 2, , 3, 4, 5, 6, 7\}$. Consider the collection of Mendelsohn triples T:

$$(0,5,4)_M, (0,4,5)_M, (0,1,4)_M, (0,4,1)_M, (1,5,2)_M, (1,5,7)_M, (1,3,5)_M,$$

$$(1,6,5)_M, (4,6,7)_M, (4,3,6)_M, (4,7,2)_M, (4,2,3)_M, (0,7,6)_M, (0,6,3)_M,$$

$$(0,2,7)_M, (0,3,2)_M, (1,2,6)_M, (2,5,6)_M, (1,7,3)_M, (3,7,5).$$

Then (S, T, P) is a maximal covering of D_8 with padding P, two disjoint copies of D_2, where $A(P) = \{(0,4), (4,0), (1,5), (5,1)\}$.

Case 1b. Suppose $v \equiv 2 \pmod 6$, $v \neq 8$, say $v = 6t + 2$ where $t \geq 2$, P is two disjoint copies of D_2. Let

$$S = \{0, 1, \ldots, v-6, a, b, c, d, e\} = \{0, 1, \ldots, 6t-4, a, b, c, d, e\}.$$

Theorem DPC.D (continued 1)

Proof for $v \equiv 2$ (mod 6), continued. Case 1a. Suppose $v = 8$ and P is two disjoint copies of D_2. Let $S = \{0, 1, 2, , 3, 4, 5, 6, 7\}$. Consider the collection of Mendelsohn triples T:

$$(0,5,4)_M, (0,4,5)_M, (0,1,4)_M, (0,4,1)_M, (1,5,2)_M, (1,5,7)_M, (1,3,5)_M,$$

$$(1,6,5)_M, (4,6,7)_M, (4,3,6)_M, (4,7,2)_M, (4,2,3)_M, (0,7,6)_M, (0,6,3)_M,$$

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Theorem DPC.D (continued 2)

Proof for $v \equiv 2 \pmod 6$, continued. Consider the collection of Mendelsohn triples:

$$\{(0, 2+i, 3t-1-i)_M \mid i = 0, 1, \ldots, t-2\}$$

$$\cup\{(0, 4t+i, t-1-i)_M \mid i = 0, 1, \ldots, t-3\}$$

$$\cup\{(0,1,a)_M, (0, 4t-3, b)_M, (0, 4t-2, c)_M, (0, 4t-1, d)_M, (0, 6t-4, e)_M\}$$

$$\cup\{(a,b,e)_M, (a,e,b)_M, (a,e,c)_M, (a,d,e)_M,$$

$$(a,c,d)_M, (c,e,d)_M, (b,c,d)_M, (b,d,c)_M\}.$$

These triples, along with their images under the powers of the permutation $(0, 1, \ldots, 6t-4)(a)(b)(c)(d)(e)$, form a covering of D_v with the padding P, two disjoint copies of D_2, where
$A(P) = \{(a,e), (e,a), (d,c), (c,d)\}$. □