Chapter 1. Steiner Triple Systems

Note. Combinatorial design theory has its earliest beginnings in puzzles, such as "magic squares" in which the numbers $1, 2, \ldots, n^2$ are arranged in a square $n \times n$ array such that the sum of the numbers in each row, each column, and both diagonals are all the same. Some of these were known in antiquity; the Wikipedia webpage on magic squares has a good deal of history and some academic references. In addition to the recreational mathematics setting, combinatorial designs started to see applications in the design of experiments in the 20th century. I tell the following motivational story in my online presentation "What the Hell do Graph Decompositions have to do with Experimental Designs," available online as a PowerPoint presentation. Consider a laboratory machine that can be used to compare samples from different mice. To keep the machine balanced while running, it must have three samples in it. However, it is impossible to keep the machine calibrated from one run to the next, so that the only way to compare to mice is to run their samples together in the machine. Since the properties of this artificial machine are exactly those needed to motivate our consideration of Steiner triple systems, we call it the "Motivationtron-3000." For efficiency, we wish to compare a certain number of samples to each other by running each pair in the machine exactly once. For how many samples can this be done and how? The facts that we are comparing pairs of samples and the machine requires that a triple of samples to be ran together implies that solving the problem for n samples is equivalent to finding a Steiner triple system of order n. This is illustrated in the figure from my talk given below. We will see in this chapter that a Steiner triple system of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$. We will establish this be giving constructions of such Steiner triple systems for each order (thus showing *how* the samples should be grouped into triples in runs of the machine).

1.1. The Existence Problem

Note. In this section, we define Steiner triple systems, give necessary conditions for their existence, and claim a sufficient condition for their existence (we show the conditions to be sufficient in the other sections of this chapter).

Definition. A *Steiner triple system* if an ordered pair (S, T), where S is a finite set of *points* or *symbols*, and T is a set of 3-element subsets of S called *triples*, such that each pair of distinct elements of S occurs together in exactly on triple or T. The *order* of a Steiner triple system (S, T) is the size of set S, denoted $|S|$. We sometimes denoted a Steiner triple system of order v as a $STS(v)$.

Example 1.1.1. **(a)** Consider $S = \{1\}$ and $T = \varnothing$. Notice that every pair of distinct elements of S (of which there are none) occur together in exactly one triple of T (of which there are none) vacuously! So (S, T) forms a $STS(1)$.

(b) Consider $S = \{1, 2, 3\}$ and $T = \{\{1, 2, 3\}\}$. First, notice that T is a set of sets (where each element of T is a set with 3 elements), by definition. Now every pair of elements of S certainly occurs exactly once in an element of T. This result holds "trivially" because its validity is very obvious (whereas the result in part (a) holds vacuously because there is nothing in the definition of a Steiner triple system to check in that case; these two examples illustrate the subtle difference between a result which holds vacuously and one which holds trivially). So (S, T) forms a $STS(3)$.

(c) Consider $S = \{1, 2, 3, 4, 5, 6, 7\}$ and $T = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$. There is a total of $\binom{7}{2} = \frac{(7)(6)}{2} = 21$ pairs of distinct elements of S. There are 7 triple in T and each includes 3 pairs, for a total of 21 pairs of elements of S (though this counting argument does not alone allow us to conclude that all the pairs are different). We can go through these 21 pairs by hand and confirm that the set of triples really does contain each pair exactly one. So (S, T) forms a $STS(7)$. You might notice a suggestive pattern in the collection of triples. We'll take advantage of this pattern in certain constructions of Steiner triple systems.

(d) Consider $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let set T include the triples:

$\{1, 2, 3\}$	$\{1, 4, 7\}$	$\{1, 5, 9\}$	$\{1, 6, 8\}$
$\{4, 5, 6\}$	$\{2, 5, 8\}$	$\{2, 6, 7\}$	$\{2, 4, 9\}$
$\{7, 8, 9\}$	$\{3, 6, 9\}$	$\{3, 4, 8\}$	$\{3, 5, 7\}$

Here again there are $\binom{9}{2} = \frac{(9)(8)}{2} = 36$ pairs of distinct elements of S, and the triples of T contain $(12)(3) = 36$ pairs of points, as needed. We can check the 36 pairs of points and confirm that each pair occurs in exactly one triple, so that (S, T) forms a $STS(9)$. You probably notice that each of the three columns of triples contain the numbers 1 through 9 exactly once. This property is not a coincidence and results in this $STS(9)$ being a Kirkman triple system (a topic covered in Chapter 5).

Note. The definition of a Steiner triple system given above lacks any sort of direct geometric interpretation. However, we can can treat the pairs of elements as edges of a graph and the triples as 3-cycles (or "triangles") in a graph. A Steiner triple system of order n is then an edge-decomposition of the complete graph K_n into triangles. For an introduction to graph theory, see my online notes for Introduction to Graph Theory (MATH 4347/5347). As an example, the complete graph on 7 vertices, K_7, is given in Figure 1.2. In Figure 1.3 the concept of a 3-cycle decomposition of a complete graph (K_7 in Figure 1.3) is illustrated.

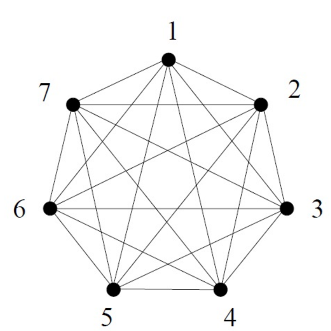


Figure 1.2: The complete graph K_7.

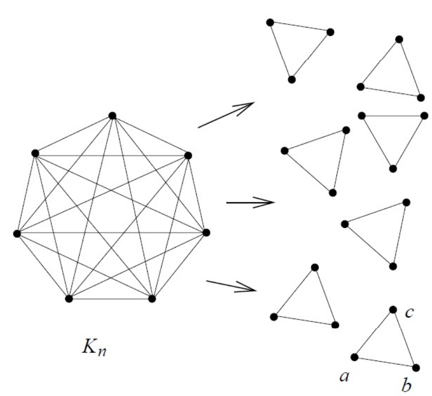
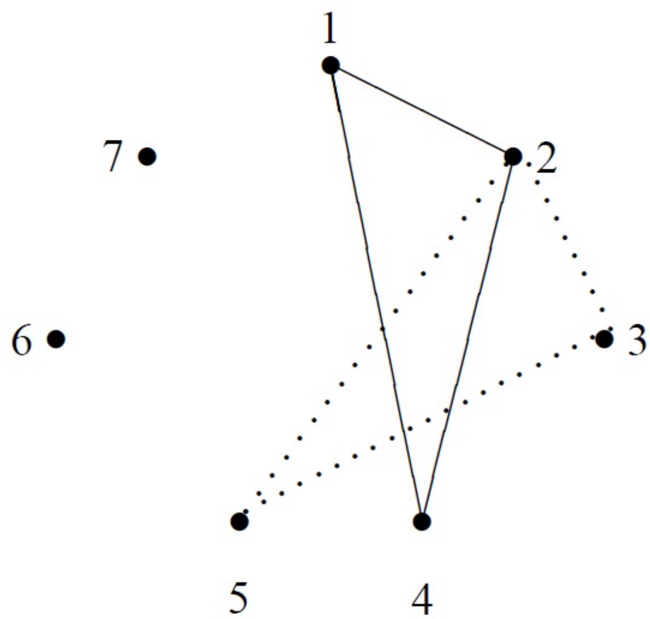


Figure 1.3: Steiner triple system and a decomposition of K_n into triangles.

Note 1.1.A. If we consider Example 1.1.1(c), where a $STS(7)$ was given, in terms of this new interpretation then we see that the 7 triples result by taking the triangle with vertices 1, 2, and 4 (which corresponds to the triple $\{1, 2, 4\}$ in T) and rotating it around the set of seven vertices as follows:



We see that the rotation can be accomplished by the mapping of the vertices $i \mapsto i + 1$ for $i = 1, 2, 3, 4, 5, 6$ and $7 \mapsto 1$. This type of mapping (called a "cyclic" mapping) will be of special interest in Section 1.7. Cyclic Steiner Triple Systems.

Note. Lindner and Rodger say (page 3) that Steiner triple systems were apparently first defined by W. S. B. Wool-House in 1844 in the *Lady's and Gentlemen's Diary* as "Prize Question 1733." The problem was ultimately solved by Thomas P. Kirkman (March 31, 1806–February 4, 1895) in "On a Problem of Combinations," *Cambridge and Dublin Mathematics Journal*, **2** (1847), 191–204. Ironically, Steiner triple systems are named for Jakob Steiner (March 18, 1796–April 1, 1863), a Swiss mathematician working in Berlin most of his career, who gave necessary conditions for their existence and published it in "Combinatorische Aufgabe," *Journal für die Reine und angewandte Mathematik* (*Crelle's Journal*), **45** (1853), 181–182. The strange dates on the necessary conditions of Steiner and the sufficiency of Kirkman are explained by a lack of communication between mainland Europe and the British Isles at the time—this likely results from fallout from the argument between Newton and Leibniz over who deserves the credit for inventing/discovering calculus.

Jakob Steiner (1796–1863)

Thomas P. Kirkman (1806–1895)

These images are from the MacTutor History of Mathematics Archive (accessed 5/8/2022).

Note. We now state and prove necessary conditions for the existence of a $STS(v)$. This is the 1853 result of Jakob Steiner.

Lemma 1.1.A. If a Steiner triple system of order v exists then $v \equiv 1$ or 3 (mod 6).

Note. In fact the necessary conditions of Lemma 1.1.A are sufficient, as shown in Kirkman in 1847. Hence, the following theorem holds. We will show sufficiency more than once (but not in the way that Kirkman did). We'll establish sufficiency in Section 1.2. $v \equiv 3 \pmod 6$: The Bose Construction and Section 1.3. $v \equiv 1 \pmod 6$: The Skolem Construction, and again in Section 1.7. Cyclic Steiner Triple Systems.

Theorem 1.1.3. A Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$.

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