1.4. $v \equiv 5 \pmod 6$: The $6n+5$ Construction

Note. In this section, we define a pairwise balanced design and see that it is a generalization of the idea of a Steiner triple system. We give a construction of a pairwise balanced design of order $6n+5$ which has one block of size 5 and all other blocks of size 3. This special structure will by useful in Section 1.5. Quasigroups with Holes and Steiner Triple Systems.

Definition. A *pairwise balanced design*, or PBD, is an ordered pair (S, B), where S is a finite set of symbols, and B is a collection of subsets of S called *blocks*, such that each pair of distinct elements of S occurs together in exactly one block of B. The *order* of the PBD is $|S|$.

Note. A Steiner triple system is a pairwise balanced design in which each block has size 3. Notice that in a general PBD, there is not a requirement that the blocks are all the same size. The next example of a PBD is a special case of the type of PBD we construct in general below.

Example 1.4.1(b). Let $S = \{1, 2, \ldots, 11\}$ and let B be the set of blocks:

$\{1,2,3,4,5\}$	$\{2,6,9\}$	$\{3,7,8\}$	$\{4,8,11\}$
$\{1,6,7\}$	$\{2,7,11\}$	$\{3,9,10\}$	$\{5,6,8\}$
$\{1,8,9\}$	$\{2,8,10\}$	$\{4,6,10\}$	$\{5,7,10\}$
$\{1,10,11\}$	$\{3,6,11\}$	$\{4,7,9\}$	$\{5,9,11\}$

Then (S, B) is a PBD with one block of size 5 and the rest of size 3.

Note. We now describe the "$6n + 5$" construction. It is similar to the Bose construction and is based on an idempotent commutative quasigroup. Let $v = 6n + 5$ where $n \in \mathbb{N}$ and let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2n + 1$, where $Q = \{1, 2, \ldots, 2n + 1\}$ (which is known to exist by Exercise 1.2.3(a,iii)). Let α be the permutation $(1)(2, 3, 4, \ldots, 2n + 1)$ and let S be the set $S = \{\infty_1, \infty_2\} \cup (\{1, 2, \ldots, 2n + 1\} \times \{1, 2, 3\})$. Let B contain the following blocks:

Type 1: We have the Type 1 block $\{\infty_1, \infty_2, (1, 1), (1, 2), (1, 3)\}$.

Type 2: For $1 \le i \le n$ we have the Type 2 blocks $\{\infty_1, (2i, 1), (2i, 2)\}$, $\{\infty_1, (2i, 3), ((2i)\alpha, 1)\}$, $\{\infty_1, ((2i)\alpha, 2), ((2i)\alpha, 3)\}$, $\{\infty_2, (2i, 2), (2i, 3)\}$, $\{\infty_2, ((2i)\alpha, 1), ((2i)\alpha, 2)\}$, $\{\infty_2, (2i, 1), ((2i)\alpha^{-1}, 3)\}$.

Type 3: For $1 \le i < j \le 2n+1$ we have the Type 3 blocks $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), ((i \circ j)\alpha, 1)\}$.

In Exercises 1.4.6 and 1.4.7, it is to be shown that this construction actually gives a PBD of order $6n + 5$.

Note. In Figure 1.6 below the three different types of blocks given above are illustrated. The Type 1 block is clear. The Type 3 triples are self explanatory (though the they are slightly complicated by the use of permutation $\alpha = (1)(2, 3, \ldots, 2n + 1)$). The Type 2 blocks contain either ∞_1 or ∞_2 and are represented in terms of thin lines in Figure 1.6 (which join pairs of points which are in triples with ∞_1) and thick lines in Figure 1.6 (which join pairs of points which are in triples with ∞_2).

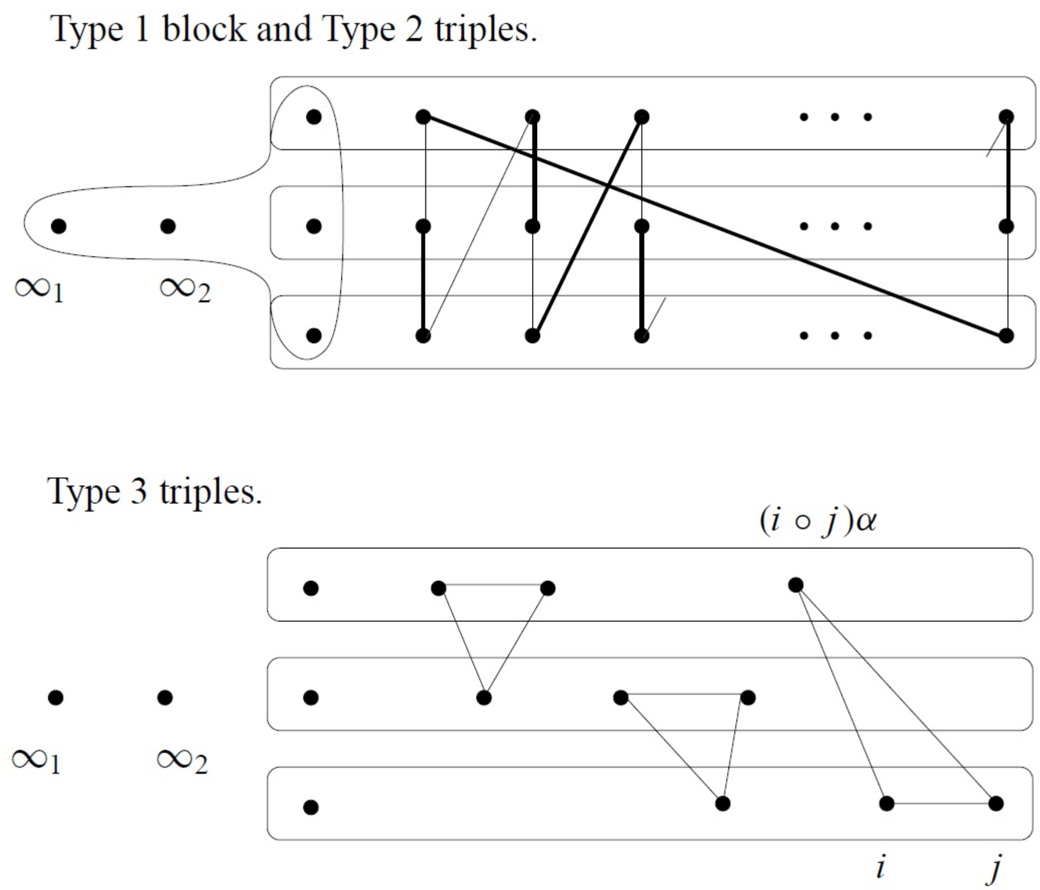

Figure 1.6: The $6n + 5$ Construction.

Example 1.4.3. We illustrate the $6n + 5$ construction to find a PBD of order 11 (in which case $n = 1$) with one block of size 5 and the rest of size 3. We need an idempotent commutative quasigroup $(Q, \circ)$ or order $2n + 1 = 2(1) + 1 = 3$ and use the following:

$\circ$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Define $\alpha = (1)(2,3)$ (so that $1\alpha = 1$, $2\alpha = 3$, and $3\alpha = 2$). Then $S = \{\infty_1, \infty_2\} \cup (\{1,2,3\} \times \{1,2,3\})$, and B contains the following blocks:

Type 1: We have the Type 1 block $\{\infty_1, \infty_2, (1,1), (1,2), (1,3)\}$.

Type 2: $\{\infty_1, (2,1), (2,2)\}$, $\{\infty_1, (2,3), (3,1)\}$, $\{\infty_1, (3,2), (3,3)\}$, $\{\infty_2, (2,2), (2,3)\}$, $\{\infty_2, (3,1), (3,2)\}$, $\{\infty_2, (2,1), (3,3)\}$.

Type 3: For $i = 1$ and $j = 2$ we get the triples:

$$\{(1,1), (2,1), ((1 \circ 2), 2)\} = \{(1,1), (2,1), (3,2)\}$$

$$\{(1,2), (2,2), ((1 \circ 2), 3)\} = \{(1,2), (2,2), (3,3)\}$$

$$\{(1,3), (2,3), ((1 \circ 2)\alpha, 1)\} = \{(1,3), (2,3), (3\alpha, 1)\} = \{(1,3), (2,3), (2,2)\}.$$

For $i = 1$ and $j = 3$ we get the triples:

$$\{(1,1), (3,1), ((1 \circ 3), 2)\} = \{(1,1), (3,1), (2,2)\}$$

$$\{(1,2), (3,2), ((1 \circ 3), 3)\} = \{(1,2), (3,2), (2,3)\}$$

$$\{(1,3), (3,3), ((1 \circ 3)\alpha, 1)\} = \{(1,3), (3,3), (2\alpha, 1)\} = \{(1,3), (3,3), (3,2)\}.$$

For $i = 2$ and $j = 3$ we get the triples:

$$\{(2,1), (3,1), ((2 \circ 3), 2)\} = \{(2,1), (3,1), (1,2)\}$$

$$\{(2,2), (3,2), ((2 \circ 3), 3)\} = \{(2,2), (3,2), (1,3)\}$$

$$\{(2,3), (3,3), ((2 \circ 3)\alpha, 1)\} = \{(2,3), (3,3), (1\alpha, 1)\} = \{(2,3), (3,3), (1,1)\}.$$

We have that (S, B) is a PBD of order 11, as desired.