1.5. Quasigroups with Holes and Steiner Triple Systems

Note. In this section, we define a quasigroup with holes and show that such a structures exist of order $2n$ for all $n \geq 3$ (in Theorem 1.5.5). We then use these to construct Steiner triple systems.

Definition. A *quasigroup with holes* H, where $H = \{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$, is a quasigroup of order $2n$, $(Q,\circ)$, where $Q = \{1,2,\ldots,2n\}$, in which for each $h \in H$, $(h,\circ)$ is a subquasigroup of $(Q,\circ)$ of order 2.

Example 1.5.1. Notice that each $h \in H$ is a pair of elements of Q. So for each $h = \{x,y\}$, there are only two choices for the subquasigroup:

$\circ$	x	y
x	x	y
y	y	x

$\circ$	x	y
x	y	x
y	x	y

The quasigroup below left is of order 6 holes $H = \{\{1,2\},\{3,4\},\{5,6\}\}$, and the quasigroup below right is of order 8 with holes $H = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$. Notice that each is commutative.

∘	1	2	3	4	5	6
1	**1**	**2**	5	6	3	4
2	**2**	**1**	6	5	4	3
3	5	6	**3**	**4**	1	2
4	6	5	**4**	**3**	2	1
5	3	4	1	2	**5**	**6**
6	4	3	2	1	**6**	**5**

∘	1	2	3	4	5	6	7	8
1	**1**	**2**	5	6	7	8	3	4
2	**2**	**1**	8	7	3	4	6	5
3	5	8	**3**	**4**	2	7	1	6
4	6	7	**4**	**3**	8	1	5	2
5	7	3	2	8	**5**	**6**	4	1
6	8	4	7	1	**6**	**5**	2	3
7	3	6	1	5	4	2	**7**	**8**
8	4	5	6	2	1	3	**8**	**7**

Note 1.5.A. We will show that commutative quasigroups exist for all even orders of at least 6. We use a construction based on modifying a pairwise balanced design (see Section 1.4). The construction possibly requires renaming of the symbols of the PBD(v), where $v \neq 5$, at most one block is of size 5 and the rest of the blocks are size 3. We use the symbols $\{1, 2, \ldots, v\}$ and want the PBD(v) to contain the triples $\{1, 2, v\}$, $\{3, 4, v\}$, $\ldots$, $\{v-2, v-1, v\}$. We illustrate the process to accomplish this in the next example, but leave the general result as an exercise (Exercise 1.5.A). In Exercise 1.5.10, a simpler construction based on direct products is to be given, but it only applies to orders of quasigroups which are 2 modulo 4.

Example 1.5.2. Consider the STS(9) on the set $S = \{1, 2, 3\} \times \{1, 2, 3\}$ of Example 1.2.4. The set of triples are

$$T = \{\{(1,1),(1,2),(1,3)\},\{(2,1),(2,2),(2,3)\},\{(3,1),(3,2),(3,3)\},$$

$$\{(1,1),(2,1),(3,2)\},\{(1,2),(2,2),(3,3)\},\{(1,3),(2,3),(3,1)\},$$

$$\{(1,1),(3,1),(2,2)\},\{(1,2),(3,2),(2,3)\},\{(1,3),(3,3),(2,1)\},$$

$$\{(2,1),(3,1),(1,2)\},\{(2,2),(3,2),(1,3)\},\{(2,3),(3,3),(1,1)\}.$$

We rename an arbitrary symbol, say $(1,1)$, as $v = 9$. We can arbitrarily rename another symbol 1, say symbol $(1,2)$. We see that $\{(1,1),(1,2),(1,3)\}$ is a triple in the STS, so we *must* rename the symbol $(1,3)$ as 2 so that the triple $\{1,2,9\}$ is in the renamed PBD, as desired. Again we can arbitrarily choose something to rename 3, say $(2,1)$. Since $\{(1,1),(2,1),(3,2)\}$ is a triple then we *must* rename $(3,2)$ as 4 so that the triple $\{3,4,9\}$ is in the renamed PBD. Next, we (arbitrarily, under the constraints of not reusing an ordered pair) rename $(3,1)$ as 5, observe that $\{(1,1),(3,1),(2,2)\}$ is a triple, and rename $(2,2)$ as 6 so that the triple $\{5,6,9\}$ is in the renamed PBD. Finally, we rename $(2,3)$ as 7, observe that $\{(2,3),(3,3),(1,1)\}$ is a triple, and rename $(3,3)$ as 8, so that the triple $\{7,8,9\}$ is in the renamed PBD. Since all symbols of the original STS(9) have been renamed is such a way that the renamed (triple) system contains triples $\{1,2,9\}$, $\{3,4,9\}$, $\{5,6,9\}$, and $\{7,8,9\}$, then we have accomplished the construction (and all other triples are determined by the new names).

Exercise 1.5.A. For any PBD(v), where $v \neq 5$, with at most one block is of size 5 and the rest of the blocks are size 3, the symbols of the PBD can be renamed using the symbols $\{1,2,\dots,v\}$ in such a way that the renamed PBD(v) contains the triples $\{1,2,v\}$, $\{3,4,v\}$, $\dots$, $\{v-2,v-1,v\}$.

Note. We now have the equipment to prove our main result of this section.

Theorem 1.5.5. For all $n \geq 3$, there exists a commutative quasigroup of order $2n$ with holes $H = \{\{1,2\},\{3,4\},\dots,\{2n-1,2n\}\}$.

Example 1.5.6. We now illustrate Theorem 1.5.5 by constructing first a commutative quasigroup with holes H of order 8 using a $STS(9)$. We use the STS(9) of Example 1.2.4 that we translated into new symbols in Example 1.5.2. The triples in terms of the symbols $\{1, 2, \ldots, 9\}$ are (notice the first four triples which are as described in Note 1.5.A):

$$B = \{\{1,2,9\}, \{3,4,9\}, \{5,6,9\}, \{7,8,9\}, \{3,6,7\}, \{1,6,8\},$$
$$\{1,4,7\}, \{1,3,5\}, \{4,5,8\}, \{2,5,7\}, \{2,3,8\}, \{2,4,6\}\}.$$

The triples containing 9 (the first four triples in set B) are used in relation to the holes and so are not explicitly needed in the construction. We use the remaining triples to define $\circ$, as given in the proof of Theorem 1.5.5 in step (a) of the construction of $(Q, \circ)$. For example, triple $\{3, 6, 7\}$ gives $3 \circ 6 = 6 \circ 3 = 7$, $3 \circ 7 = 7 \circ 3 = 6$, and $6 \circ 7 = 7 \circ 6 = 3$. This gives the following commutative partial quasigroup:

$\circ$	1	2	3	4	5	6	7	8
1			5	7	3	8	4	6
2			8	6	7	4	5	3
3	5	8			1	7	6	2
4	7	6			8	2	1	5
5	3	7	1	8			2	4
6	8	4	7	2			3	1
7	4	5	6	1	2	3		
8	6	3	2	5	4	1		

We complete the quasigroup $(Q, \circ)$ by filling in the holes (as in the proof of Theorem 1.5.5 in step (a)) using using one of the styles of quasigroups of order 2 in Example 1.5.1.

Next, we construct a commutative quasigroup of order $2n = 10$ using a PBD of order $2n+1 = 11 \equiv 5 \pmod 6$. We use the PBD(11) from Example 1.4.1(b) but we rename the symbols, as described in Note 1.5.A, to get the triples $\{1,2,11\}$, $\{3,4,11\}$, $\{5,6,11\}$, $\{7,8,11\}$, $\{9,10,11\}$, $\{1,4,7\}$, $\{1,6,10\}$, $\{2,3,6\}$, $\{2,4,9\}$, $\{2,5,7\}$, $\{2,8,10\}$, $\{3,7,10\}$, $\{4,5,10\}$, $\{4,6,8\}$, $\{6,7,9\}$, and the block of size five of $b = \{1,3,5,8,9\}$. We use the latin square of order 5 from Example 1.2.2 (with the symbols renamed):

$\otimes$	1	3	5	8	9
1	1	8	3	9	5
3	8	3	9	5	1
5	3	9	5	1	8
8	9	5	1	8	3
9	5	1	8	3	9

Ignoring the five triples containing 11, using the remaining ten triples to define $\circ$ for a pair of elements not in block b (step (b)), and using block b to define $\circ$ for the 20 pairs of *distinct* elements b (step (c); these values of $\circ$ are given in bold faced fonts), we get the following partial quasigroup:

$\circ$	1	2	3	4	5	6	7	8	9	10
1			**8**	7	**3**	10	4	**9**	**5**	6
2			6	9	7	3	5	10	4	8
3	**8**	6			**9**	2	10	**5**	**1**	7
4	7	9			10	8	1	6	2	5
5	**3**	7	**9**	10			2	**1**	**8**	4
6	10	3	2	8			9	4	7	1
7	4	5	10	1	2	9			6	3
8	**9**	10	**5**	6	**1**	4			**3**	2
9	**5**	4	**1**	2	**8**	7	6	**3**		
10	6	8	7	5	4	1	3	2		

Again, we complete the quasigroup $(Q, \circ)$ by filling in the holes (step (a)) using using one of the styles of quasigroups of order 2 in Example 1.5.1.

Note. Theorem 1.5.5 shows how we can use a Steiner triple system or a pairwise block design consisting of a block of size five and all other blocks as triple to construct a commutative quasigroup with holes. We now reverse the direction of the construction and use a commutative quasigroup of order $2n$ with holes H to produce a STS$(6n+1)$, a STS$(6n+3)$, and a PBD$(6n+5)$.

Note. The Quasigroup with Holes Construction of a STS and a PBD. Let $(\{1, 2, \ldots, 2n\}, \circ)$ be a commutative quasigroup of order $2n$ with holes H. Then

(a) $(\{\infty\} \cup (\{1, 2, \ldots, 2n\} \times \{1, 2, 3\}), B)$ is an STS$(6n+1)$, where B is defined by:

(1) for $1 \le i \le n$ let B_i contains the triples in a STS(7) on the symbols $\{\infty\} \cup (\{2i-1, 2i\} \times \{1, 2, 3\})$ and let $B_i \subseteq B$, and

(2) for $1 \le i \ne j \le 2n$, $\{i, j\} \notin H$ place the triples $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, and $\{(i, 3), (j, 3), (i \circ j, 1)\}$ in B (these are like the Type 2 triples in the Bose construction of Section 1.2).

(b) $(\{\infty_1, \infty_2, \infty_3\} \cup (\{1, 2, \ldots, 2n\} \times \{1, 2, 3\}), B')$ is a STS$(6n+3)$, where B' is defined by replacing (1) in (a) with:

(1′) for $1 \le i \le n$ let B_i' contain the triples in an STS(9) on the symbols $\{\infty_1, \infty_2, \infty_3\} \cup (\{2i-1, 2i\} \times \{1, 2, 3\})$ in which $\{\infty_1, \infty_2, \infty_3\}$ is a triple, and let $B_i' \subseteq B'$, and

and we follow step (2) of (a).

(c) $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (\{1, 2, \ldots, 2n\} \times \{1, 2, 3\}), B'')$ is a PBD$(6n+5)$ with one block of size 5, the rest of size 3, where B'' is defined by replacing (1) in (a) with:

(1″) for $1 \leq i \leq n$ let B_i'' contain the blocks in a PBD(11) on the symbols $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (\{2i-1, 2i\} \times \{1, 2, 3\})$ in which $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ is a block, and let $B_i'' \subseteq B''$,

and we follow step (2) of (a).

It follows from Exercise 1.1.4 that (a) and (b) produce Steiner triple systems, and it follows from Exercise 1.4.6 that (c) produces the desired pairwise balanced design. Figure 1.7 gives a way to visualize the construction.

Example 1.5.11. We now construct a STS(31). Since $31 \equiv 1 \pmod 6$, then we use step (a) of The Quasigroup with Holes Construction. First (part (1)), we take the blocks B_i of a STS(7) on the symbols $\{\infty\} \cup \{2i-1, 2i\} \times \{1, 2, 3\}$ for $1 \leq i \leq 5$ (since $31 = 6(5) + 1$, we have $n = 5$). We are not stranger to a STS(7) (see Section 1.1. The Existence Problem), so we take:

$$B_i = \{\{\infty, (2-1, 1), (2i, 1)\}, \{\infty, (2i-1, 2), (2i, 2)\}, \{\infty, (2i-1, 3), (2i, 3)\},$$
$$\{(2i-1, 1), (2i-1, 2), (2i-1, 3)\}, \{(2i-1, 1), (2i, 2), (2i, 3)\},$$
$$\{(2i, 1), (2i-1, 2), (2i, 3)\}, \{(2i, 1), (2i, 2), (2i-1, 3)\}\} \text{ for } 1 \leq i \leq 5.$$

This gives us a total of 35 triples. Next (part (2)), we need a commutative quasigroup with holes H of order $2n = 10$, and we use the one above in Example 1.5.6.

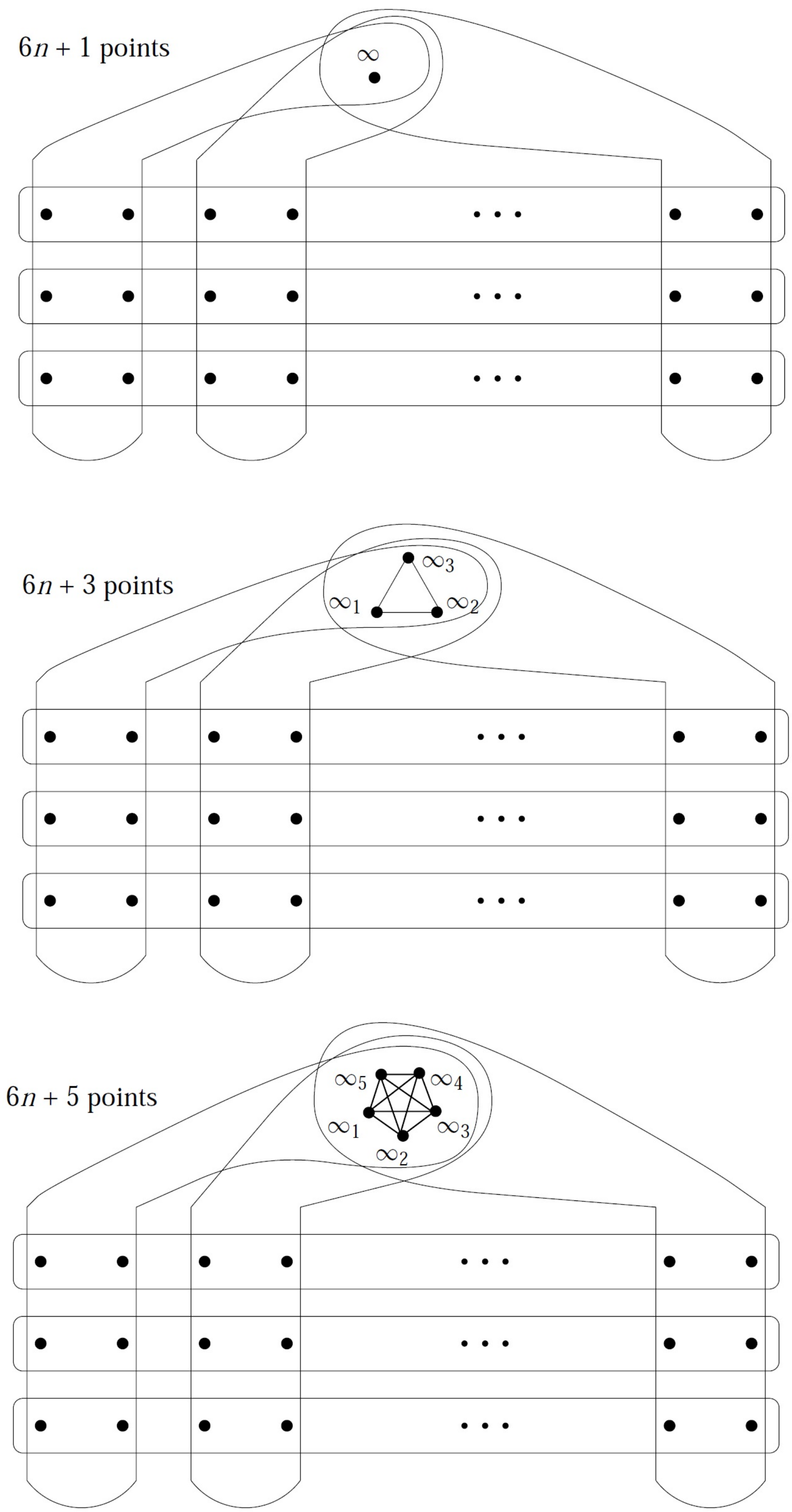

Figure 1.7: The Quasigroup with Holes Construction.

Since a STS(31) has $\binom{31}{2}/3 = 155$ triples (so we need another 120 in part (2)), we only consider the triples containing the pairs of symbols (i) $(4,1)$ and $(5,2)$, (ii) $(5,1)$ and $(6,2)$, and (iii) ∞ and $(8,3)$. For (i), we look for need a triple of the type $\{(i,1),(j,1),(i \circ j,2)\}$ so we have $i = 4$ and $i \circ j = 4 \circ j = 5$ (notice that $\{i,j\} = \{4,5\} \notin H$). We see from the commutative quasigroup with holes of Example 1.5.6 that we need to take $j = 10$, so that the triple is $\{(4,1),(10,1),5,2)\}$. For (ii), if we look for a triple of the type $\{(i,1),(j,1),(i \circ j,2)\}$ with $i = 5$ and $i \circ j = 5 \circ j = 6$, but we see from Example 1.5.6 that this implies that j is either 5 or 6 (depending on how the holes are filled) and so $\{i,j\} \in H$. This means that one of the triples based on a STS(7) already contains this pair; in fact it is contained in one of the triples in B_3 (where $2i-1 = 5$ and $2i = 6$), namely $\{(5,1),(6,2),(6,3)\}$. For (iii), we know that triples containing the symbol ∞ are addressed in part (1) with a STS(7); in fact it is contained in one of the triple in B_4 (where $2i = 8$), namely $\{\infty,(7,3),(8,3)\}$. We leave the other 117 triples as an exercise(!).

Example 1.5.13. We now construct a STS(35). Since $35 \equiv 5 \pmod 6$, then we use step (c) of The Quasigroup with Holes Construction. Since $35 = 6(5)+1$, we have $n = 5$ and we need a PBD$(2n+1) =$ PBD(11) with one black of size 5 and all other blocks as triple; an example of which is given in Example 1.4.1(b). From $(1'')$, we let

$$B_i'' = \{\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\},\{\infty_1,(2i-1,1),(2i,1)\},\{\infty_1,(2i-1,2),(2i,2)\},$$

$$\{\infty_1,(2i-1,3),(2i,3)\},\{\infty_2,(2i-1,1),(2i,2)\},\{\infty_2,(2i,1),(2i,3)\},$$

$$\{\infty_2,(2i-1,2),(2i-1,3)\},\{\infty_3,(2i,1),(2i-1,2)\},\{\infty_3,(2i,2),(2i-1,3)\},$$

$$\{\infty_3,(2i-1,1),(2i,3)\},\{\infty_4,(2i-1,2),(2i,3)\},\{\infty_4,(2i-1,1),(2i-1,3)\},$$
$$\{\infty_4,(2i,1),(2i,2)\},\{\infty_5,(2i-1,1),(2i-1,2)\},\{\infty_5,(2i,1),(2i-1,3)\},$$
$$\{\infty_5,(2i,2),(2i,3)\} \text{ for } 1 \le i \le 5.$$

For step (2) As in Example 1.5.11, for step (2) we only consider the triples containing the pairs of symbols (i) $(4,1)$ and $(5,2)$, (ii) $(5,1)$ and $(6,2)$, and (iii) ∞_1 and $(8,3)$. For (i), we look for need a triple of the type $\{(i,1),(j,1),(i\circ j,2)\}$ and, exactly as in Example 1.5.11, the resulting triple is $\{(4,1),(10,1),5,2)\}$. For (ii), similar to Example 1.5.11, because 5 and 6 are both in a hole of the quasigroup, we have that one of the triples from $(1'')$ contains this pair; in fact it is contained in one of the triples in B_3 (where $2i-1=5$ and $2i=6$), namely $\{\infty_2,(6,2),(6,3)\}$. For (iii), we know that triples containing the symbol ∞_1 are addressed in part $(1'')$; in fact it is contained in one of the triple in B_4 (where $2i=8$), namely $\{\infty_1,(7,3),(8,3)\}$.

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