1.6. The Wilson Construction

Note. In this section, we give direct constructions for a STS(v) for all $v \equiv 1$ or 3 (mod 6). After setting up the appropriate equipment, the constructions are rather brief. The results are originally due to Richard M. Wilson and appeared in "Some Partitions of all Triples into Steiner Triple Systems," in Hypergraph Seminar, Ohio State Univ. 1972, *Lecture Notes in Mathematics*, Vol. 411, pp. 267–277 Berlin: Springer-Verlag, Berlin (1974).

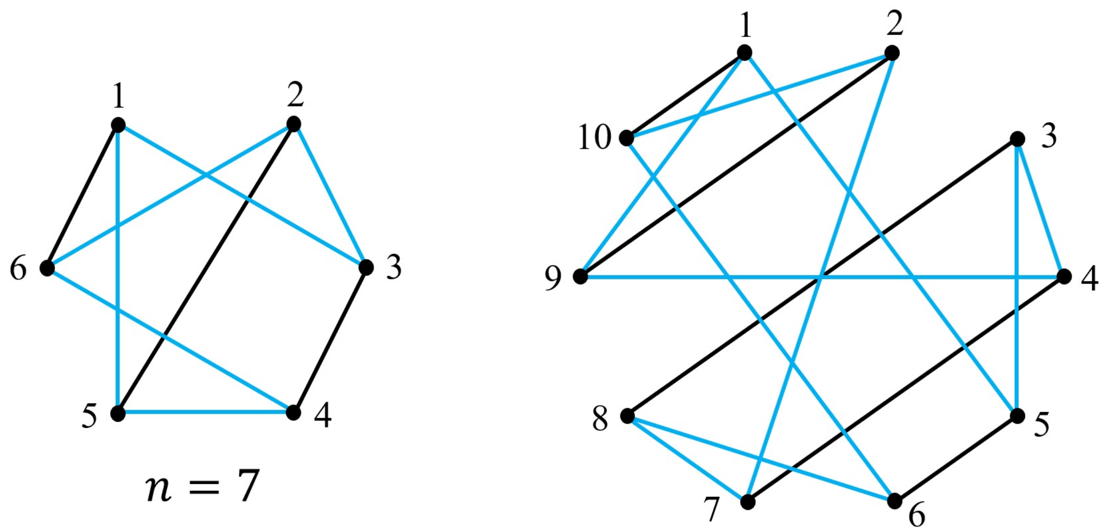
Definition. A 1-*factor* of a graph G is a set of pairwise disjoint edges which partition the vertex set. A 2-*factor* of G is a 2-regular spanning subgraph. A 1-*factorization* of G is a set of 1-factors which partition the edge set of G.

Note. A more general idea of an r-factor (that is, an r-regular spanning subgraph) is covered in Introduction to Graph Theory (MATH 4347/5347); see my online notes for this class on Section 2.2. Edge Colorings. The next definition is specific to the material presented here and is not a standard topic in a graph theory class.

Definition. The *deficiency graph* of additive group $(\mathbb{Z}_n, +)$, where $n \equiv 1$ or 5 (mod 6), is the graph $G = (V, E)$ with vertex set $V = \mathbb{Z}_n \setminus \{0\}$ and edge set $E = \{\{x, -x\}, \{x, -2x\} \mid x \in \mathbb{Z}_n \setminus \{0\})\}$.

Note 1.6.A. Below are examples of the deficiency graphs in the cases $n = 7$ and $n = 11$ the edges of the form $\{x, -x\}$ are given in black and the edges of the form

$\{x, -2x\}$ are in blue. In these graphs, as is the case in general, are 3-regular. Each has a 1-factorization into three 1-factors. One of these 1-factors is given by the black edges, and the other two are given by the blue edges (the blue edges form an even length cycle which can be decomposed into two 1-factors by taking every-other edge in each).

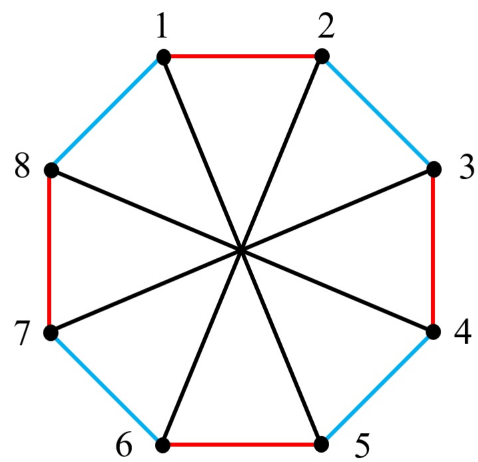


$n = 7$ $n = 11$

Definition. A *wheel* is a graph consisting of a cycle of even length and a 1-factor in which the edges join the opposite vertices of the cycle. The edges of the 1-factor are called *spokes*. **This definition of "wheel" is only valid in this book!!!**

Note. In a graph theory class, you will see a wheel defined differently. It will include a central vertex that is adjacent to each of the vertices in the cycle (and the cycle need not be of even length). The term "spokes" then applies to the edges joining the central vertex to the vertices on the cycle. See my online notes for Introduction to Graph Theory (MATH 4347/5347) on Section 2.1. Vertex Colorings.

Example 1.6.1. Following Lindner and Rodger's definition of "wheel," we see that a wheel has a 1-factorization into three 1-factors. One of the 1-factors consists of the spokes of the wheel. The other two 1-factors make up the even cycle. The wheel on 8 vertices is as follows, where the three 1-factors are given by black, blue, and red edges.

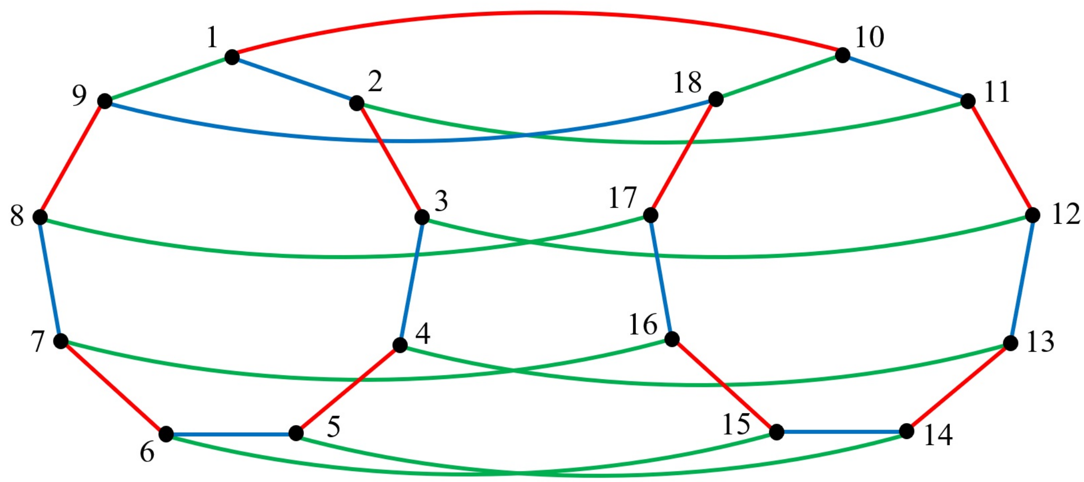


Definition. A *biwheel* is a graph consisting of the union of two vertex disjoint cycles C_1 and C_2 of the same length n and a 1-factor F consisting of n edges such that: (i) each edge in F is incident with one vertex in C_1 and one vertex in C_2, and (ii) the mapping $\alpha : V(C_1) \to V(C_2)$ defined by $x\alpha = y$ if and only if $\{x, y\} \in F$ is a graph isomorphism of C_1 onto C_2.

Lemma 1.6.A. Biwheels can be 1-factored into three 1-factors.

Example 1.6.2. Consider the biwheel with C_1 and C_2 as cycles of length nine. Label the vertices of C_1 as $1, 2, \ldots, 9$ and label the vertices of C_2 as $10, 11, \ldots, 18$, as given below. Let α the the mapping $V(C_1)$ to $V(C_2)$ defined as $\alpha(i) = i + 9$.

Then the edges in the three one factors are as shown below where the edges in $P_1 \cup P_1\alpha \cup F_1$ are blue, the edges in $P_2 \cup P_2\alpha \cup F_2$ are red, and the edges in $P_3 \cup P_3\alpha \cup F_3$ are green (by the way, 1-factorizations are commonly considered in the setting of edge colorings of graphs).



Here we have taken alternating edges of C_1 as blue and red, until we get "stuck" at the last edge. We then color that edge (which join vertices 1 and 9) green. This gives the minimum number of green edges in C_1 (this is not necessary, but it works!). Then, all edges in F are colored green except for those with ends of 1 or 9. Due to the symmetry of the 1-factors in C_1 and C_2 (given in general by isomorphism α), we are guaranteed that the two edges in F with ends of 1 or 9 can be colored blue or red.

Note. Next, we present the Deficiency Graph Algorithm, which shows that a deficiency graph has a 1-factorization using three 1-factors. This factorization will then be used in Wilson's Construction to give a Steiner triple system of order v for all $v \equiv 1$ or $3 \pmod 6$.

Note. The Deficiency Graph Algorithm. Let $m \equiv 1$ or 5 (mod 6) and let $G = (V, E)$ be the deficiency graph $(\mathbb{Z}_m, +)$ defined above. Let $F = \{\{x, -x\} \mid x \in \mathbb{Z}_m \setminus \{0\}\}$ and $T = \{\{x, -2x\} \mid x \in \mathbb{Z}_m \setminus \{0\}\}$. Notice that F is a 1-factor of $(\mathbb{Z}_m, +)$. As stated in Note 1.6.A, T is a spanning 2-factor of $(\mathbb{Z}_m, +)$.

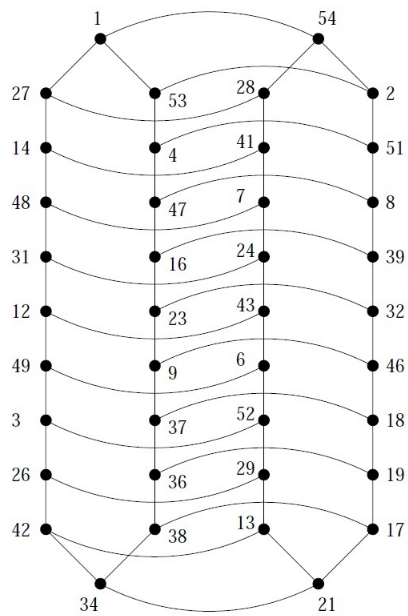
(1) Since T is a 2-factor of G, then it can be decomposed into vertex disjoint cycles. This follows from a result presented in Graph Theory 1 (MATH 5340); see my online notes for this class on Section 2.4. Decompositions and Coverings (notice Veblen's Theorem, Theorem 2.7). We consider each such cycle C in turn.

(2) If C is a cycle of even length then either each pair of opposite vertices in C is of the form $\{x, -x\}$ or there exists another cycle C^* such that $x \in C$ if and only if $-x \in C^*$, as is to be shown in Exercise 1.6.9. Whichever case holds, let $F(C)$ be the set of edges of F which cover the vertices of C (or of $C \cup C^*$) and form the wheel $C \cup F(C)$ (or the biwheel $C \cup C^* \cup F(C)$).

(3) If C is a cycle of odd length then there exists a cycle C^* such that $x \in C$ if and only if $-x \in C^*$ (again, by Exercise 1.6.9). Let $F(C)$ be the set of edges of F which cover the vertices of $C \cup C^*$ and form the biwheel $C \cup C^* \cup F(C)$.

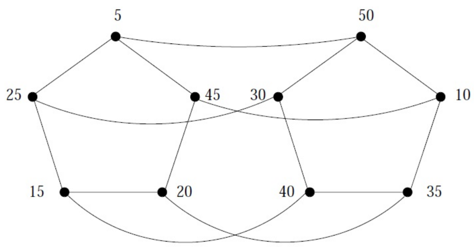
(4) Steps (2) and (3) partition the deficiency graph into vertex disjoint wheels and biwheels. Now there is a 1-factorization of each wheel (see Example 1.6.1) and biwheel (see Lemma 1.6.A) into three 1-factors each. Union these 1-factors together to get a 1-factorization of the deficiency graph into three 1-factors.

Example 1.6.3. Consider the deficiency graph $(\mathbb{Z}_{55}, +)$. It has the three components given below.

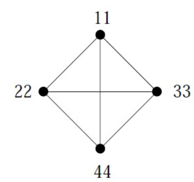


1	53	53	4	1	54
4	47	47	16	53	2
16	23	23	19	27	28
9	37	37	36	4	51
36	38	38	34	14	41
34	42	42	26	47	8
26	3	3	49	48	7
49	12	12	31	16	39
31	48	48	14	31	24
14	27	27	1	23	32
54	2	2	51	12	43
51	8	8	39	9	46
39	32	32	46	49	6
46	18	18	19	37	18
19	17	17	21	3	52
21	13	13	29	36	19
29	52	52	6	26	29
6	43	43	24	38	17
24	7	7	41	42	13
41	28	28	54	34	21

This is a biwheel, where the cycle on the left includes edges of the form $\{x, -2x\}$ (where vertex labels are reduced modulo 55). It starts at vertex 1, then produces in order $-2(1) = -2 \equiv 53 \pmod{55}$, $-2(53) = -106 \equiv 4 \pmod{55}$, $-2(4) = -8 \equiv 47 \pmod{55}$, $-2(47) - 94 \equiv 16 \pmod{55}$, etc. Notice that it returns to vertex 1 after picking up a total of 20 vertices (so it is a cycle of even length). The cycle on the right includes edges of the form $\{x, -2x\}$, starting at vertex 54 (notice that $-1 \equiv 54 \pmod{55}$; this is the reason for starting with vertex 54). This gives the vertices in order of produces in order $-2(54) = -108 \equiv 2 \pmod{55}$, $-2(2) = -4 \equiv 51 \pmod{55}$, $-2(51) = -102 \equiv 8 \pmod{55}$, $-2(8) = -16 \equiv 39 \pmod{55}$, $-2(39) - 78 \equiv 32 \pmod{55}$, etc. It too returns to its starting vertex of 54 after picking up a total of 20 vertices (and so is also a cycle of even length). The remaining edges in set F of the biwheel are of the form $\{x, -x\}$. This biwheel has a 1-factorization into three 1-factors by Lemma 1.6.A. The one factors are given in columns of the table. Notice that the alternating-edge approach has been used since the cycles are of even length (this is suggested in the way the first two columns are subdivided half-way down).



5	45	45	20	25	5
20	15	15	25	30	50
50	10	10	35	45	10
35	40	40	30	20	35
25	30	5	50	15	40



11	33	33	44	11	44
44	22	22	11	22	33

On the left of the biwheel the (odd length) cycle starts with vertex 5 and then picks up edges of the form $\{x, -2x\}$, reducing labels modulo 55 as we go, returning to vertex 5. In the (odd length) cycle on the right of the biwheel, we start with vertex 50 (notice that $-5 \equiv 50 \pmod{55}$). The remaining edges are of the form $\{x, -x\}$ and so are in F. This biwheel has a 1-factorization into three 1-factors by Lemma 1.6.A. The one factors are given in columns of the table. Since the cycles here are of odd length, we use the trickier approach illustrated in Example 1.6.1 to find the three 1-factors. In the complete graph on the right, we have a cycle of length 4 (even) in which each opposite pair of vertices is of the form $\{x, -x\}$ (in this case, the pairs are 11,44 and 22,33). This is case (2) in the Deficiency Graph Algorithm and we see that the complete graph is actually a wheel, which we know to have a 1-factorization (as in Example 1.6.1), which is given in the three columns. Now, combining all the edges in each of the three columns gives the three 1-factors in the 1-factorization of the deficiency graph $(\mathbb{Z}_{55}, +)$.

Note. We are now in a position to introduce Wilson's Construction of Steiner triple systems. In his construction the Type 1 triples foreshadow the approach of difference methods used at length in the next section on cyclic Steiner triple systems.

Note. Wilson's Construction of Steiner Triple Systems. Let $v \equiv 1$ or 3 (mod 6) and set $S = \{\infty_1, \infty_2\} \cup \mathbb{Z}_{v-2}$. Notice that $v - 2 \equiv 1$ or 5 (mod 6). Define the following triples.

Type 1. $T^* = \{\{x, y, z\} \mid x + y + z \equiv 0 \pmod{v-2}\}$, where x, y, z are distinct elements in $\mathbb{Z}_{v-2} \setminus \{0\}\}$.

Type 2. The 2-element subsets of $\mathbb{Z}_{v-2} \setminus \{0\}$ not covered by a Type 1 triple are precisely the 2-element subsets of the form $\{x, -x\}$ and $\{x, -2x\}$, as is to be shown in Exercise 1.6.8. Let F_0, F_1, and F_2 be a 1-factorization of the deficiency graph of $(\mathbb{Z}_{v-2}, +)$ and define

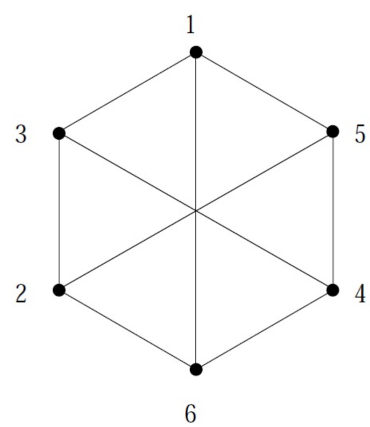
$$\begin{aligned} T_0 &= \{\{0, x, y\} \mid \{x, y\} \in F_0\}, \\ T_1 &= \{\{\infty_1, x, y\} \mid \{x, y\} \in F_1\}, \text{ and} \\ T_2 &= \{\{\infty_2, x, y\} \mid \{x, y\} \in F\}. \end{aligned}$$

Type 3. $\{0, \infty_1, \infty_2\}$.

Set $T = T^* \cup T_0 \cup T_1 \cup T_2 \cup \{\{0, \infty_1, \infty_2\}\}$. Then (S, T) is an STS(v). To see this claim, notice that every pair x, y where $x, y \in \mathbb{Z}_{v-2} \setminus \{0\}$ occurs (exactly once) either in a triple of T^* or in one of the 1-factors F_0, F_1, or F_2. If the pair occurs in one of the 1-factors, then it appears in (exactly one) of the triples of $_0$, T_1, or T_2.

A pair of the form $0, x$ where $x \in \mathbb{Z}_{v-2} \setminus \{0\}$ appears in a triple of T_0. A pair of the form ∞_1, x (respectively ∞_2, x) where $x \in \mathbb{Z}_{v-2} \setminus \{0\}$ appears in a triple of T_1 (respectively, T_2). A pair involving two of $0, \infty_1, \infty_2$ appears in the triple of Type 3. This covers all possible pairs of elements of $S = \{\infty_1, \infty_2\} \cup \mathbb{Z}_{v-2}$, so we indeed have a STS(v), as claimed.

Example 1.6.4. Consider the case $v = 9$. By the Wilson Construction, we take $S = \{\infty_1, \infty_2\} \cup \mathbb{Z}_7$. Since $1 + 2 + 4 \equiv 3 + 5 + 6 \equiv 0 \pmod 7$ (and these are the only distinct nonzero elements of $\mathbb{Z}_7$ satisfying this condition), then the Type 1 triples are $\{1, 2, 4\}$ and $\{3, 5, 6\}$. The deficiency graph of $\mathbb{Z}_7$ (a wheel, in fact) and a 1-factorization of it into three 1-factors F_1, F_2, F_3 is:



F_0	F_1	F_2
1 3	2 3	1 6
2 6	4 6	2 5
4 5	1 5	3 4

The Type 2 triples are then (say): $\{0, 1, 3\}$, $\{0, 2, 6\}$, $\{0, 4, 5\}$, $\{\infty_1, 2, 3\}$, $\{\infty_1, 4, 6\}$, $\{\infty_1, 1, 5\}$, $\{\infty_2, 1, 6\}$, $\{\infty_2, 2, 5\}$, $\{\infty_2, 3, 4\}$. The Type 3 triple is the $\{0, \infty_1, \infty_2\}$. Notice that we have $12 = \binom{9}{2}/3$, as expected.

Revised: 8/21/2022