1.8. The $2n + 1$ and $2n + 7$ Constructions

Note. In this section, we give two constructions which will allow us to construct Steiner triple systems recursively. The two constructions allow us to use a STS(n) to construct a STS($2n + 1$) and a STS($2n + 7$). These constructions will also be useful when we consider the intersection problem for Steiner triple systems in Section 8.2. The General Intersection Problem.

Definition. Let n be odd. A 1-*factorization* of K_{n+1} is a pair (X, F) where F is a partition of K_{n+1} into n 1-factors with vertex set X.

Note 1.8.A. Let $(Q, \circ)$ be an idempotent commutative quasigroup of order n, where $Q = \{1, 2, \ldots, n\}$(n is odd here and such a quasigroup exists by Exercise 1.2.3(a,iii)). For each $i \in Q$ let $F_i = \{\{i, n + 1\}\} \cup \{\{a, b\} \mid a \circ b = b \circ a = i\}$. We now show that $F = \{F_1, F_2, \ldots, F_n\}$ is a 1-factorization of K_{n+1} with vertex set $X = \{1, 2, \ldots, n\}$. As $\{a, b\}$ ranges over all possible unordered pairs of distinct elements of Q, we see that all edges of K_{n+1} of the form $\{a, b\}$ (since $(Q, \circ)$ is commutative, order does not matter) are present in F (we can find which F_i contains $\{a, b\}$ using $\circ$; it's $F_{a\circ b} = F_{b\circ a}$). An edge of the form $\{a, n + 1\}$ is in F_a. Hence, all edges of K_{n+1} are present in F. Since i appears exactly once in each row and exactly once in each column of the quasigroup, then each F_i is a 1-factor of K_{n+1}.

Example 1.8.1. The technique of Note 1.8.A can be used with the following idempotent commutative quasigroup (left) to produce the following 1-factorization of K_8 (right; with the obvious notation).

$\circ$	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

$F =$

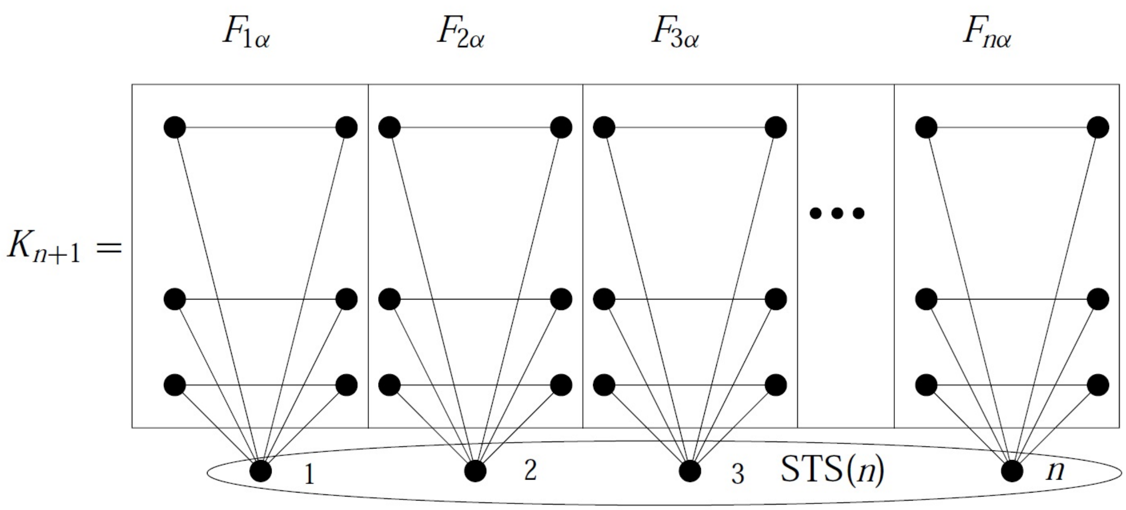
45	56	67	35	46	57	34
36	47	24	26	37	23	25
27	13	15	17	12	14	16
18	28	38	48	58	68	78
F_1	F_2	F_3	F_4	F_5	F_6	F_7

Note. The $2n+1$ Construction of a STS($2n+1$) from a STS(n).
Let (S,T) be a Steiner triple system of order n and let (X,F) be a 1-factorization of K_{n+1} with vertex set X, where $X \cap S = \varnothing$. Let $S^* = S \cup X$ and define a collection of triples T^* as follows:

(1) $T \subseteq T^*$ (that is, T^* contains all of T).

(2) Let α be any one-to-one (injective) mapping from S onto $\{1, 2, \ldots, n\}$ (so α is a bijection). For each $x \in S$ and each $\{a, b\} \in F_{x\alpha}$, place the triple $\{x, a, b\}$ in T^*

The (S^*, T^*) is a Steiner triple system of order $2n+1$ (as is to be shown in Exercise 1.8.3). An illustration of the construction is:



Example 1.8.2. We now illustrate the $2n+1$ construction to make a STS$(2n+1)$ = STS(15) from a STS(n) = STS(7) and a 1-factorization of K_8. Let (S, T) be a Steiner triple system of order 7 where $S = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Let (X, F) be the 1-factorization of $K_{n+1} = K_8$ given in Example 1.8.1 above. Define $\alpha : S \to \{1, 2, \ldots, 7\}$ as $x_i\alpha = i$ for each $\in \{1, 2, \ldots, 7\}$. By (1) of the $2n+1$ Construction, each triple of the STS(7) is in T^* (and these contain all pairs of the form $\{x_i, x_j\}$ where $i \neq j$). To illustrate (2) of the construction, suppose we want to find the triple containing the pair $\{3, 7\}$. We look for the 1-factor of F containing $\{3, 7\}$ and we see that it is $F_{x_i\alpha} = F_5$. So with $x_i\alpha = i = 5$, we take the triple $\{x_5, 3, 7\} \in T^*$. Similarly, since (in the notation of Example 1.8.1) $F_5 = \{46, 37, 12, 58\}$, we also have the triples $\{x_5, 4, 6\}$, $\{x_5, 1, 2\}$, and $\{x_5, 5, 8\}$ in T^*. Suppose we want to find the triple containing the pair $\{x_4, 2\}$. Since $x_4\alpha = 4$, then we look for 2 in the 1-factor $F_{x_4\alpha} = F_4$ and find the pair $26 = \{2, 6\}$. So we take the triple $\{x_4, 2, 6\} \in T^*$. Similarly, since (in the notation of Example 1.8.1) $F_4 = \{35, 26, 17, 48\}$, we also have the triples $\{x_4, 3, 5\}$, $\{x_4, 1, 7\}$, and $\{x_4, 4, 8\}$ in

T^*. Hence, along with the triples of the STS(7) we have the following triples in T^*:

$$\{x_1,4,5\},\{x_1,3,6\},\{x_1,2,7\},\{x_1,1,8\};\{x_2,5,6\},\{x_2,4,7\},\{x_2,1,3\},\{x_2,2,8\};$$

$$\{x_3,6,7\},\{x_3,2,4\},\{x_3,1,5\},\{x_3,3,8\};\{x_4,3,5\},\{x_4,2,6\},\{x_4,1,7\},\{x_4,4,8\};$$

$$\{x_5,4,6\},\{x_5,3,7\},\{x_5,1,2\},\{x_5,5,8\};\{x_6,5,7\},\{x_6,2,3\},\{x_6,1,4\},\{x_6,6,8\};$$

$$\{x_7,3,4\},\{x_7,2,5\},\{x_7,1,6\},\{x_7,7,8\}$$

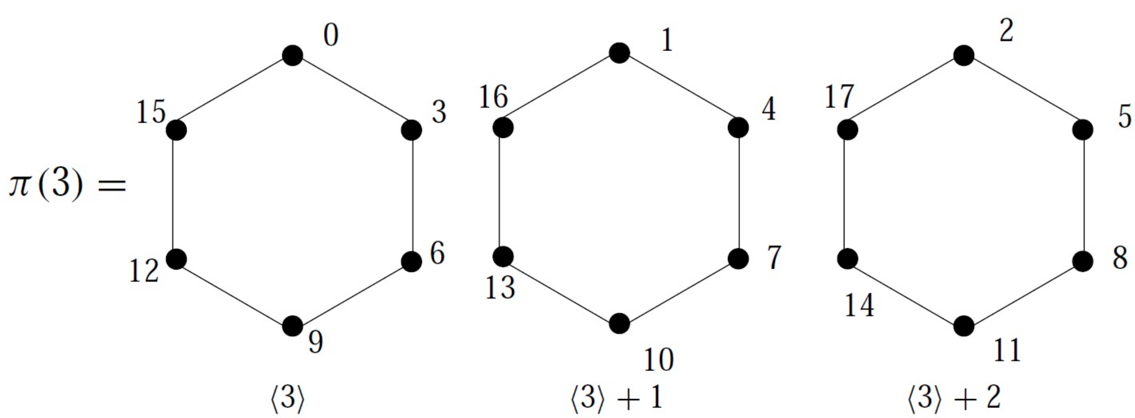
Notice that the STS(7) has $\binom{7}{2}/3 = 7$ triples and we have an additional 28 triples here, for a total of $\binom{15}{2}/3 = 35 = 7 + 28$ triples, as expected. □

Note. A cycle on vertices $x_1, x_2, \ldots, x_k$ with edges $\{x_1, x_2\}$, $\{x_2, x_3\}$, …, $\{x_{k-1}, x_k\}$, $\{x_k, x_1\}$ is denoted by any cyclic shift of the k-tuple $(x_1, x_2, \ldots, x_k)$ or $(x_1, x_k, x_{k-1}, \ldots, x_2)$. We define the *length* of an edge $\{a, b\}$ in K_n as the length of the shortest path between a and b in the cycle $(1, 2, 3, \ldots, n)$. Notice that each edge in K_n has a length of at most $\lfloor n/2 \rfloor$. This is the same idea as a "difference" associated with an edge in the technique of difference methods for the construction of cyclic Steiner triple system; see my online notes for Graph Theory 1 (MATH 5340) on Supplement. Graph Decompositions: Triple Systems. Now consider a complete graph K_{2n} with vertex set $\mathbb{Z}_{2n}$. For $d \in \mathbb{Z}_{2n} \setminus \{0\}$ (that is, we take $0 < d < n$), denote by $\langle d \rangle$ the cycle $(0, d, 2d, 3d, \ldots)$ (where we reduce the vertex labels modulo $2n$).

Example 1.8.7. In K_{12} we have, for example, the cycles $\langle 4 \rangle = (0, 4, 8)$ and $\langle 5 \rangle = (0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7)$.

Note. If $0 < d < n$ then the set of all edges of K_{2n} of length d is a 2-factor of K_{2n} (think of taking the edge $\{0, d\}$ and then repeatedly applying the permutation $\alpha : i \mapsto i + 1 \pmod{2n}$ until the edge returns to its original position). A 2-factor is a 2-regular spanning subgraph by definition, so it can be partitioned into vertex disjoint cycles (see, for example, my online notes for Introduction to Graph Theory (MATH 4347/5347) on Section 3.1. Eulerian Circuits; notice Theorem 3.1.5). These cycles are of the form $\langle d \rangle + i$ for $0 \le i < \gcd(d, n)$ (where $\gcd(d, n)$ is the greatest common divisor of d and n; notice that if d and n are relatively prime then $\langle d \rangle$ is an n-cycle). Hence all the cycles are of the same length. We denote this collection of cycles (called a "parallel class" since the cycles are disjoint) as $\pi(d)$ or $\pi_{2n}d$. We have considered edges of length $0 < d < n$ in K_{2n}, but there are also edges of length $d = n$. These edges for a 1-factor $\langle d \rangle$ of K_{2n}.

Example 1.8.8. The parallel class $\pi(3)$ in K_{18} is:



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