Chapter 2. λ-Fold Triple Systems

Note. In this chapter, we consider collections of triples of elements of a set such that every pair of elements occurs together in λ triples. With particular attention on $\lambda = 2$ in Section 2.3, we then consider Mendelsohn triple systems in Section 2.4 which have an interpretation as decompositions of complete digraphs into directed triples. In Section 2.6, give a complete solution to the existence of λ-fold triple systems.

2.1. Triple Systems of Index $\lambda > 1$

Note. In this brief section, we define λ-fold triple systems (which, when $\lambda = 1$, correspond to Steiner triple systems), and give a few illustrative examples.

Definition. A *λ-fold triple system* (or a triple system of *index* λ) is a pair (S, T), where S is a finite set and T is a collection of 3-element subsets of S called *triples* such that each pair of distinct elements of S belong to exactly λ triples of T. The *order* of a λ-fold triple system (S, T) is $|S|$.

Note. A Steiner triple system is a 1-fold triple system. Just as a Steiner triple system of order v is equivalent to to a (edge disjoint) decomposition of the complete graph K_v into triangles (that is, into K_3's or 3-cycles), a λ-fold triple system is equivalent to a (edge disjoint) decomposition of the λ-fold complete graph λK_v into triangles. In this way, we have a geometric interpretation of λ-fold triple systems. This is illustrated for $\lambda = 2$ (and $v = 7$) in Figure 2.1 below.

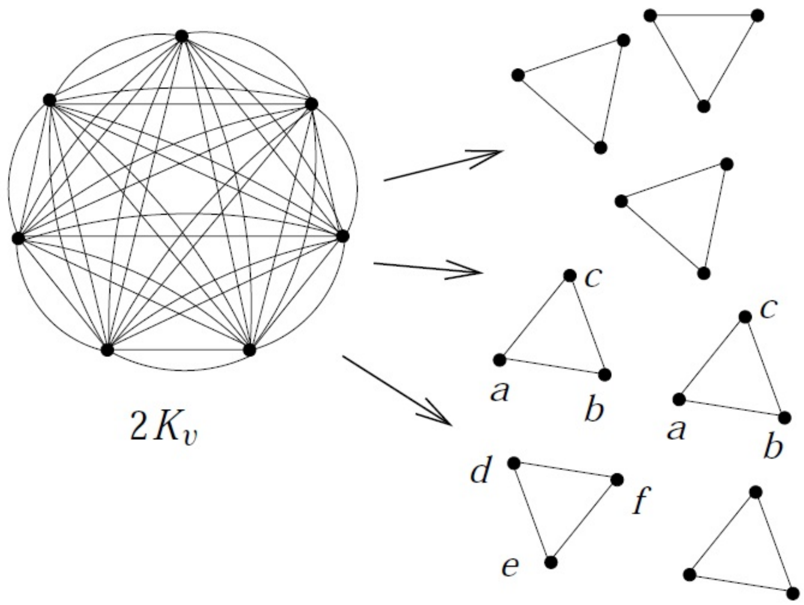

Figure 2.1: 2-fold triple system.

Example 2.1.1. We now give an example of a 3-fold triple system of order $v = 5$. We take $S = \{1, 2, 3, 4, 5\}$ and the T as the collection of triples T is:

$$\{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\},$$

$$\{3, 4, 1\}, \{4, 5, 1\}, \{4, 5, 2\}, \{5, 1, 2\}, \{5, 1, 3\}.$$

Notice that no triples are repeated in T. This is not a restriction of a λ-fold triple system. In fact, we have defined T as a "collection" of triples instead of a "set" of triples. By definition, elements of a set are not repeated. However, in a *multiset* elements can be repeated, so we could have defined T as a "multiset of triples."

Example 2.1.2. A 4-fold triple system of order 4 on the set $S = \{1, 2, 3, 4\}$ is given by the collection of triples T including

$$\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{2, 3, 4\}.$$

Notice that each triple is repeated twice.

Note. Since we impose no restriction on the repetition of triples, we can easily deal with certain order triple systems. For example, we can find a 2-fold triple system of order $v = 7$ by taking two copies of each triple in a $STS(7)$ (this is partially what is done in the text book in Example 2.1.3). More generally, we can generate a λ-fold triple system of every order $v \equiv 1$ or 3 (mod 6) by taking λ copies of each triple in a $STS(v)$. We will see in Section 2.6. λ-Fold Triple Systems in General that we only need to address the existence problem for $\lambda = 1, 2, 3,$ and 6 (see Exercise 2.6.7). Before we address any $\lambda > 1$, we need to consider idempotent commutative quasigroups of odd order (we know that idempotent commutative quasigroups of odd order exist by Exercise 1.2.3(a,iii)). This is covered in the next section.

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