2.2. The Existence of Idempotent Latin Squares

Note. In this section, we give a technique for constructing an idempotent latin square of order n from an idempotent latin square of order $n-1$. Since we know that such latin square exist for all odd n by Exercise 1.2.3(a,iii), this will allow us to show that such latin square exist for all even $n > 2$.

Note. Recall that a latin square of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \ldots, n\}$ such that each row and each column of the array contains each of the symbols in $\{1, 2, \ldots, n\}$ exactly once.

Definition. A *transversal* T of a latin square of order n on the symbols $\{1, 2, \ldots, n\}$ is a set of n cells, exactly one cell from each row and each column, such that each of the symbols in $\{1, 2, \ldots, n\}$ occurs in a cell of T.

Example 2.2.1. Let L be the latin square:

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

Some of the transversals of L include: $T_1 = \{(1,1), (2,2), (3,3), (4,4)\}$, $T_2 = \{(1,3), (2,4), (3,1), (4,2)\}$, and $T_3 = \{(1,1), (2,4), (3,2), (4,3)\}$.

Definition. Let $(\{1, 2, \ldots, n-1\}, \circ)$ is a quasigroup of order $n-1$ that contains a transversal T. Consider the set $\{1, 2, \ldots, n-1, n\}$ along with the binary operation $*$ where:

(1) For each $(i, j) \in T$, define $i * n = n * j = i \circ j$ and $i * j = n$.

(2) For each $(i, j) \notin T$, with $1 \le i, j \le n-1$, define $i * j = i \circ j$.

(3) Define $n * n = n$.

This technique of producing the new binary algebraic structure $\langle \{1, 2, \ldots, n-1, n\}, * \rangle$ is called *stripping the transversal* T of the quasigroup.

Example 2.2.2. Consider the quasigroup of order 4, along with the transversal T_2 given in yellow (left). By stripping T_2, we get the binary algebraic structure of order 5 (right).

$\circ$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

$*$	1	2	3	4	5
1	1	3	5	2	4
2	4	2	1	5	3
3	5	4	3	1	2
4	3	5	2	4	1
5	2	1	4	3	5

Notice that the entries in the transversal of the original quasigroup have been replaced with the new symbol 5, and each original element in a transversal cell has been move to the new far right column and the new lowest row (Step 1). The lower right entry is the new symbol 5 (Step 3). The other entries remain the same (Step 2).

Note. Stripping a transversal from a quasigroup of order $n-1$ yields a binary algebraic structure that is in turn a quasigroup of order n, as we now prove.

Lemma 2.2.A. Let $(\{1, 2, \ldots, n-1\}, \circ)$ be a quasigroup of order $n-1$. The order-n binary algebraic structure $\langle \{1, 2, \ldots, n-1, n\}, *\rangle$ that results from stripping a transversal of the quasigroup of order $n-1$ is itself a quasigroup.

Note. Since we know that an idempotent quasigroup exists of all odd orders by Exercise 1.2.3(a,iii) (in fact, in Exercise 1.2.3 the quasigroups were also commutative), then we can use the technique of stripping a transversal to show that idempotent quasigroups exist of all orders. We will have to carefully choose the transversal (so as to avoid the main diagonal of the given qusigroup), but stripping of the transversal may not preserve commutivity (which is not a concern, as the title of the section suggests).

Theorem 2.2.3. For all $n \neq 2$, there exists an idempotent quasigroup of order n.

Example 2.2.4. To illustrate the proof of Theorem 2.2.3, let $n = 5$ and consider the idempotent quasigroup of order 5 used in the modified version of Example 1.2.5 (which is based on a rearrangement and relabeling of $\mathbb{Z}_5$, as done in Exercise 1.2.3(a,iii)). We use the transversal, as in the above proof, of $T = \{(1,2),(2,3),(3,4),(4,5),(5,1)\}$. The idempotent quasigroup and transversal of order 5 is given here (left), along with the resulting idempotent quasigroup of order 6 (right).

∘	1	2	3	4	5
1	1	4	2	5	3
2	4	2	5	3	1
3	2	5	3	1	4
4	5	3	1	4	2
5	3	1	4	2	5

*	1	2	3	4	5	6
1	1	6	2	5	3	4
2	4	2	6	3	1	5
3	2	5	3	6	4	1
4	5	3	1	4	6	2
5	6	1	4	2	5	3
6	3	4	5	1	2	6

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