2.3. 2-Fold Triple Systems

Note. In this section, we give two constructions to establish the existence of λ-fold triple systems in the case $\lambda = 2$. We start by repeating the definition given in Section 2.1. Triple Systems of Index $\lambda > 1$, but specifically with $\lambda = 2$.

Definition. A 2-*fold triple system* (or a triple system of *index* 2) is a pair (S, T), where S is a finite set and T is a collection of 3-element subsets of S called *triples* such that each pair of distinct elements of S belong to exactly two triples of T. The *order* of a 2-fold triple system (S, T) is $|S|$.

Note. We will see that a 2-fold triple system of order v exists if and only if $v \equiv 0$ or 1 (mod 3). This will require that we consider two cases, $v = 3n$ and $v = 3n + 1$. In the event that $v \equiv 1$ or 3 (mod 6), we can simply take two copies of the triples of a $STS(v)$ to produce a 2-fold triple system.

Example 2.3.1. (b) An example of a 2-fold triple of order 4 on $S = \{1, 2, 3, 4\}$ has triples $\{1, 2, 4\}$, $\{1, 2, 3\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$.
(c) An example of a 2-fold triple of order 7 on $S = \{1, 2, 3, 4, 5, 6, 7\}$ with no repeated triples has the collection T as:

$$\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\},$$

$$\{1, 2, 6\}, \{2, 3, 7\}, \{3, 4, 1\}, \{4, 5, 2\}, \{5, 6, 3\}, \{6, 7, 4\}, \{7, 1, 5\}.$$

Note. We have the usual geometric interpretation of a 2-fold triple system of order v as a decomposition of $2K_v$ into edge disjoint triangles.

Definition. The *spectrum* for 2-fold triple systems is the set of integers v for which there exists a 2-fold triple system of order v.

Note. We will show that the spectrum of 2-fold triple systems is the set of all $v \equiv 0$ or $1 \pmod{3}$ in Theorem 2.3.7. The result will follow by two constructions, which we now consider.

Note 2.3.A. For $v \equiv 0 \pmod{3}$, we consider $v = 3n$. Let $(Q, \circ)$ be an idempotent quasigroup of order n and set $S = Q \times \{1, 2, 3\}$. We consider a collection of triples T of two types:

Type 1: The "Type 1" triple $\{(x, 1), (x, 2), (x, 3)\}$ occurs exactly twice in T for each $x \in Q$.

Type 2: The six Type 2 triples $\{(x, 1), (y, 1), (x \circ y, 2)\}$, $\{(y, 1), (x, 1), (y \circ x, 2)\}$, $\{(x, 2), (y, 2), (x \circ y, 3)\}$, $\{(y, 2), (x, 2), (y \circ x, 3)\}$, $\{(x, 3), (y, 3), (y \circ x, 1)\}$, $\{(y, 3), (x, 3), (y \circ x, 1)\}$ belong to T for all $x, y \in Q$ where $x \neq y$.

There are $2n$ Type 1 triples and $6\binom{n}{2}$ Type 2 triples. Figure 2.2 gives a visual representation of the construction.

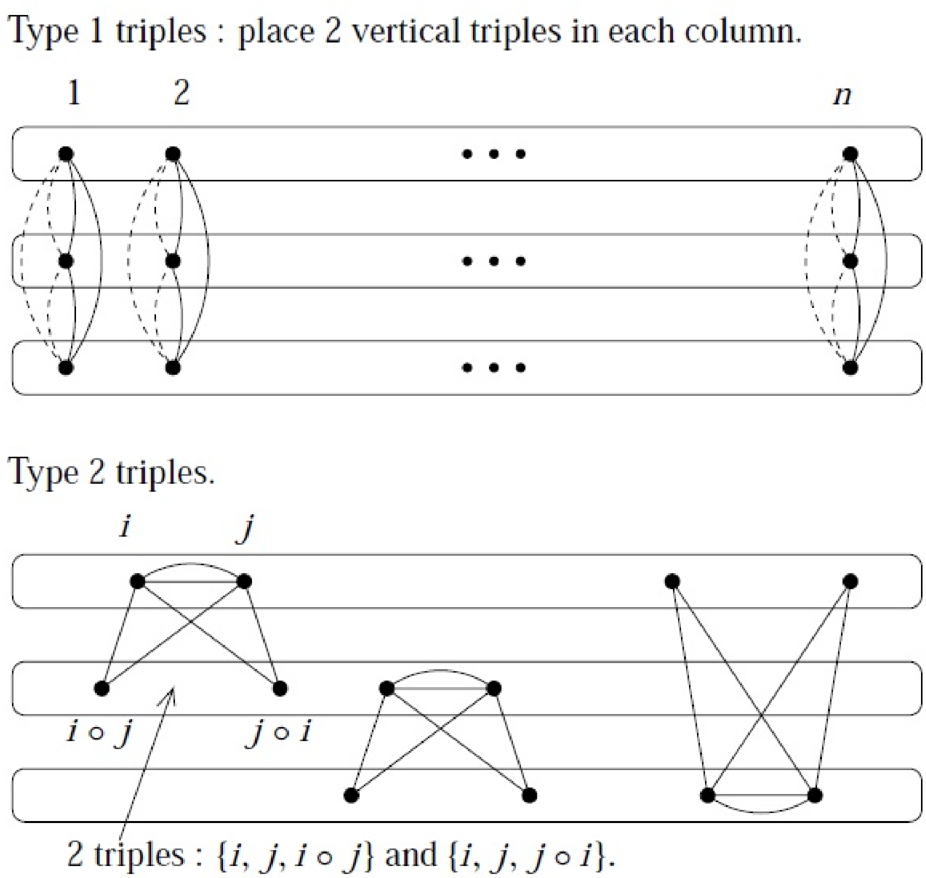

Figure 2.2: The $3n$ Construction of 2-fold triple systems.

Note 2.3.B. For $v \equiv 1 \pmod 3$, we consider $v = 3n+1$. Let $(Q, \circ)$ be an idempotent quasigroup of order n and set $S = \{\infty\} \cup (Q \times \{1,2,3\})$. We consider a collection of triples T of two types:

Type 1: The Type 1 triples $\{\infty, (x,1), (x,2)\}$, $\{\infty, (x,2), (x,3)\}$, $\{\infty, (x,1), (x,3)\}$, $\{(x,1), x, 2), (x,3)\}$ belong to T for every $x \in Q$.

Type 2: The six Type 2 triples $\{(x,1), (y,1), (x \circ y, 2)\}$, $\{(y,1), (x,1), (y \circ x, 2)\}$, $\{(x,2), (y,2), (x \circ y, 3)\}$, $\{(y,2), (x,2), (y \circ x, 3)\}$, $\{(x,3), (y,3), (y \circ x, 1)\}$, $\{(y,3), (x,3), (y \circ x, 1)\}$ belong to T for all $x, y \in Q$ where $x \neq y$.

There are $4n$ Type 1 triples and $6\binom{n}{2}$ Type 2 triples. Figure 2.3 gives a visual representation of the construction.

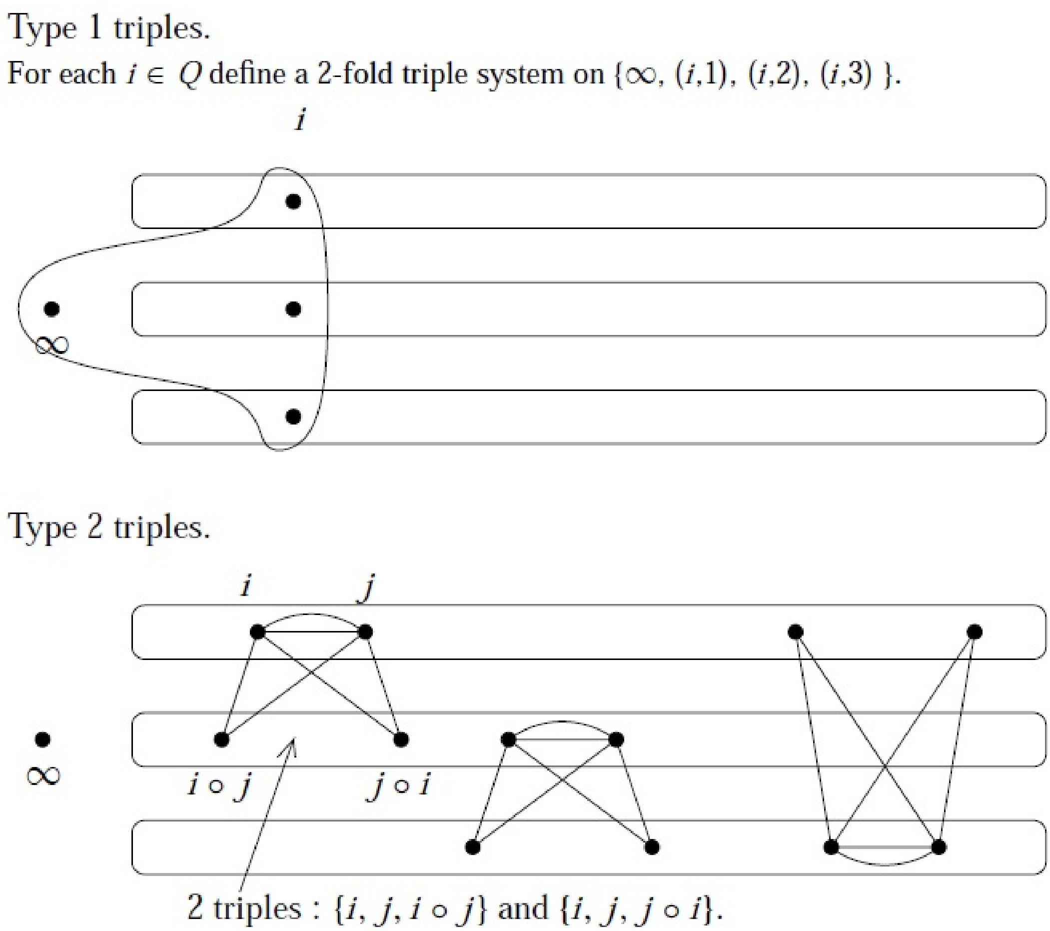

Figure 2.3: The $3n + 1$ Construction of 2-fold triple systems.

Theorem 2.3.7. The spectrum for 2-fold triple systems is precisely the of all $v \equiv 0$ or 1 (mod 3).

Example 2.3.8. We illustrate the constructions of this section using the (non-commutative) idempotent quasigroup or order 4 given below. To illustrate both constructions, we create 2-fold triple systems of orders 12 and 13.

$\circ$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

For an order 12 2-fold triple system we have the following types of blocks:

Type 1: $\{(1,1),(1,2),(1,3)\}, \{(1,1),(1,2),(1,3)\}, \{(2,1),(2,2),(2,3)\},$
$\{(2,1),(2,2),(2,3)\}, \{(3,1),(3,2),(3,3)\}, \{(3,1),(3,2),(3,3)\},$
$\{(4,1),(4,2),(4,3)\}, \{(4,1),(4,2),(4,3)\}.$

Type 2: $\{(1,1),(2,1),(3,2)\}, \{(2,1),(1,1),(4,2)\}, \{(1,1),(3,1),(4,2)\},$
$\{(3,1),(1,1),(2,2)\}, \{(1,1),(4,1),(2,2)\}, \{(4,1),(1,1),(3,2)\},$
$\{(2,1),(3,1),(1,2)\}, \{(3,1),(2,1),(4,2)\}, \{(2,1),(4,1),(3,2)\},$
$\{(4,1),(2,1),(1,2)\}, \{(3,1),(4,1),(1,2)\}, \{(4,1),(3,1),(2,2)\},$

plus two more copies of the Type 2 triples, but first with all of the second coordinates increased by 1 in the first copy, and second with all of the second coordinates (as given above) of replacing the 1's with 3's and the 2's with 1's.

For an order 13 2-fold triple system we have the following types of blocks:

Type 1: $\{\infty,(1,1),(1,3)\}, \{\infty,(1,2),(1,1)\}, \{\infty,(1,3),(1,2)\}, \{(1,1),(1,2),(1,3)\},$
$\{\infty,(2,1),(2,3)\}, \{\infty,(2,2),(2,1)\}, \{\infty,(2,3),(2,2)\}, \{(2,1),(2,2),(2,3)\},$
$\{\infty,(3,1),(3,3)\}, \{\infty,(3,2),(3,1)\}, \{\infty,(3,3),(3,2)\}, \{(3,1),(3,2),(3,3)\},$
$\{\infty,(4,1),(4,3)\}, \{\infty,(4,2),(4,1)\}, \{\infty,(4,3),(4,2)\}, \{(4,1),(4,2),(4,3)\}.$

Type 2: We use the exact same Type 2 triples as those given above in the construction of the order 12 2-fold triple system above.

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