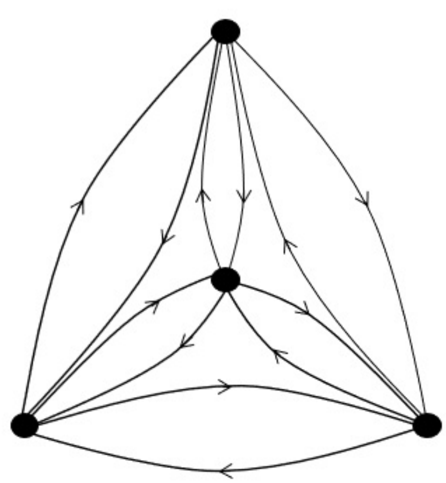
2.4. Mendelsohn Triple Systems

Note. In this section, we shift from dealing with pairs of elements of a set S to dealing with *ordered* pairs of elements. This changes the graph theoretic interpretation from edge-disjoint decompositions of graphs to arc-disjoint decompositions of digraphs.

Definition. The *complete directed graph* of order n, denoted D_n, is the digraph with n vertices in which each pair of distinct vertices are joined by two *directed edges* (in opposite directions). That is, for any pair of vertices u and v the "*arc*" from u to v and the arc from v to u are in the arc set of the digraph. We denote these arcs as (u, v) and (v, u), respectively.

Note. We represent arcs of a digraph with arrows. Digraphs are covered in Graph Theory 1 (MATH 5340) in Section 1.5. Directed Graphs, and briefly covered in Introduction to Graph Theory (MATH 4347/5347) in Section 6.2. Conservative Graphs. The complete graph on four vertices, D_4, is given in the following figure.

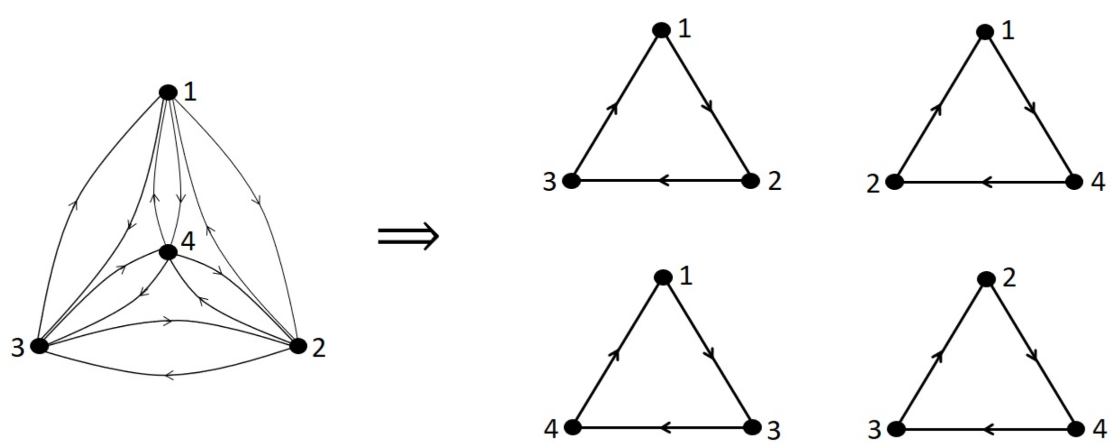


Definition. A *directed triple* is a collection of three directed edges (i.e., arcs) of the form $\{(a,b),(b,c),(c,a)\}$ where a,b,c are distinct. We denote this directed triple as (a,b,c), (b,c,a), and (c,a,b).

Note. In these notes we use the term "arc," even though Lindner and Rodger use the term "directed edge." Digraphs only play a role in this section of the text book. However, in supplemental sections we will consider additional triple systems in connection with digraphs and mixed graphs. The term "directed triple" is sometimes used to represent the collection of arcs $\{(a,b),(b,c),(a,c)\}$ where a, b, and c are distinct; this is the case when considering *directed triple systems*. However, we follow the text book's use of the term "directed triple" in these notes.

Definition. A *Mendelsohn triple system* of order n, denoted $MTS(n)$, is a pair (S,T) where T is an arc disjoint collection of directed triples which partitions the arc set of D_n on vertex set S.

Example 2.4.1. A graphical representation of a Mendelsohn triple system of order 4 is as follows:



Note. In Exercise 2.4.2 it is to be shown that a necessary condition for the existence of a $MTS(n)$ is $n \equiv 0$ or 1 (mod 3). In fact, a $MTS(n)$ exists if and only if $v \equiv 0$ or 1 (mod 6), and $n \neq 6$. This is shown in two constructions given below, and in Theorem 2.4.7. The result originally appears in Nathan S. Mendelsohn, "A Natural Generalization of Steiner Triple Systems," *Computers in Number Theory*, eds. A. O. Atkins and B. Birch, Academic Press (1971). Nendelsohn called these structures "cyclic triple systems" and the term "Mendelsohn triple system" is due to R. Mathon and A. Rosa in "A Census of Mendelsohn Triple Systems of Order Nine," *Ars Combinatoria*, **4**, 309–315 (1977).

Note. We can relate a $MTS(n)$ to a 2-fold triple system by simply replacing each arc directed triple in the $MTS(n)$ with a (undirected) triple, as is to be shown in Exercise 2.4.5.

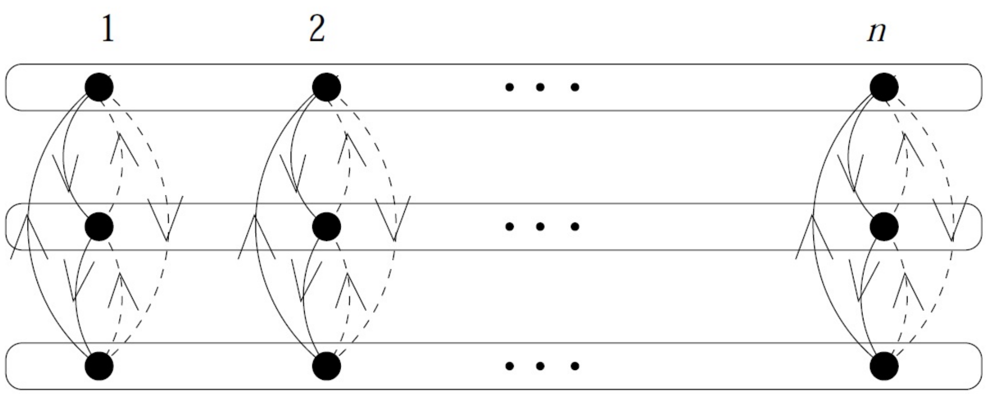
Note 2.4.A. For $v \equiv 0 \pmod 3$, $n \neq 6$, we consider $v = 3n$ where $n \neq 2$. Let $(Q, \circ)$ be an idempotent quasigroup of order n and set $S = Q \times \{1, 2, 3\}$. We consider a collection of triples T of two types:

Type 1: The "Type 1" triples $((x, 1), (x, 2), (x, 3))$ and $((x, 1), (x, 3), (x, 2))$ are in T for each $x \in Q$.

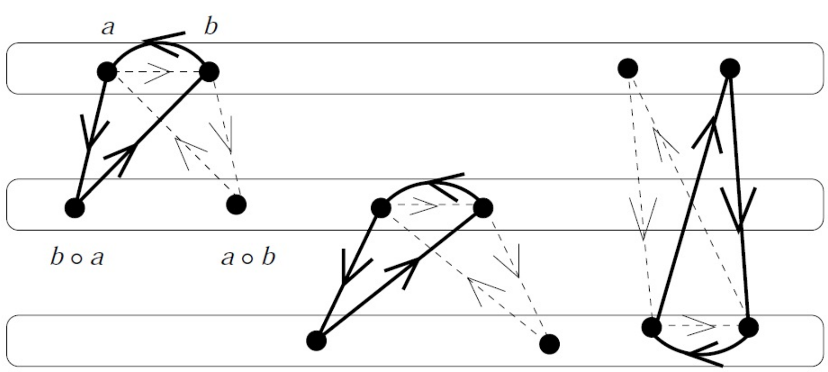
Type 2: The six Type 2 triples $((a, 1), (b \circ a, 2), (b, 1))$, $((a, 1), (b, 1), (a \circ b, 2))$, $((a, 2), (b \circ a, 3), (b, 2))$, $((a, 2), (b, 2), (a \circ b, 3))$, $((a, 3), (b \circ a, 1), (b, 3))$, $((a, 3), (b, 3), (a \circ b, 1))$ belong to T for all $a, b \in Q$ where $a \neq b$.

There are $2n$ Type 1 triples and $6\binom{n}{2}$ Type 2 triples. The next figure gives a visual representation of the construction (from pages 57 and 58).

Type 1 directed triples.



Type 2 directed triples.



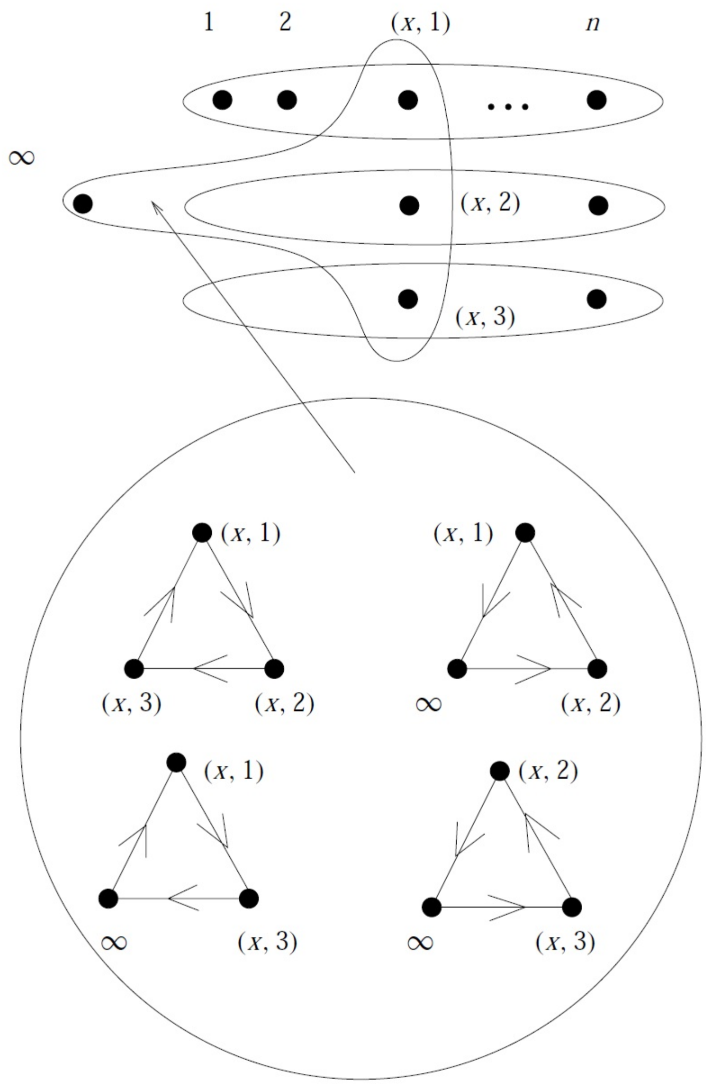
Note 2.4.B. For $v \equiv 1 \pmod 3$, $n \neq 7$, we consider $v = 3n + 1$ where $n \neq 2$. Let $(Q, \circ)$ be an idempotent quasigroup of order n and set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. We consider a collection of triples T of two types:

Type 1: The "Type 1" triples $((x,1),(x,2),(x,3))$, $((x,1),\infty,(x,2))$, $((x,1),(x,3),\infty)$, and $((x,2),\infty,(x,3))$ are in T for each $x \in Q$.

Type 2: The six Type 2 triples $((a,1),(b \circ a,2),(b,1))$, $((a,1),(b,1),(a \circ b,2))$,

$((a,2),(b\circ a,3),(b,2))$, $((a,2),(b,2),(a\circ b,3))$, $((a,3),(b\circ a,1),(b,3))$, $((a,3),(b,3),(a\circ b,1))$ belong to T for all $a,b \in Q$ where $a \neq b$.

There are $4n$ Type 1 triples and $6\binom{n}{2}$ Type 2 triples. The next figure gives a visual representation of the construction (from page 58).



Note. In Exercise 2.4.8 it is to be shown that the constructions given in Notes 2.4.A and 2.4.B actually yield Mendelsohn triple systems. A $MTS(7)$ can be found by taking a $STS(7)$ and simply replace each triple with two oppositely oreinted directed triples. A $MTS(6)$ does not exist and we follow the lead of Lindner and Rodger and appeal to Mendelsohn's original paper for the justification of this. These observations combine to give the following.

Theorem 2.4.7. The spectrum for Mendelsohn triple systems is precisely the set of all $n \equiv 0$ or $1 \pmod 3$, and $n \neq 6$.

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