2.5. $\lambda = 3$ and 6

Note. In this section we give constructions, based on idempotent quasigroups, for 3-fold and 6-fold triple systems.

Note 2.5.A. For v odd, let $(Q, \circ)$ be an idempotent commutative quasigroup of order v (which exists by Exercise 1.2.3(a,iii), in which such quasigroups are created by rearranging the Cayley table of $\mathbb{Z}_{2n+1}$). Let $T = \{\{a, b, a \circ b\} \mid a < b \in Q\}$ (here by "$a < b$" we mean the usual inequality on elements of $\{0, 1, 2, \ldots, 2n\}$ when treated as elements of $\mathbb{Z}$; though elements of $\mathbb{Z}_{2n+1}$ are actually equivalence classes on $\mathbb{Z}$ and cannot be ordered in the usual sense of the term). We'll see below that (Q, T) is a 3-fold triple system of order v. There are $0+1+2+\cdots+(v-1) = \dfrac{(v-1)v}{2}$ such triples.

Note 2.5.B. For any $v \neq 2$, let $(Q, \circ)$ be an idempotent quasigroup of order v by Theorem 2.2.3. Let $T = \{\{a, b, a \circ b\} \mid a, b \in Q, a \neq b\}$. We'll see below that (Q, T) is a 6-fold triple system of order v. There are $v(v-1)$ such triples.

Note. In Exercise 2.5.9 it is to be shown (using Exercise 2.5.5) that the constructions given in Notes 2.5.A and 2.5.B actually yield 3-fold and 6-fold triple systems of order v, respectively. We therefore have the following.

Theorem 2.5.7. The spectrum for 3-fold triple systems is precisely the set of all odd integers $v \geq 1$. The spectrum for 6-fold triple systems is precisely the set of all positive integers $v \neq 2$.

Examples 2.5.1 and 2.5.8. Consider the idempotent commutative quasigroups of orders 7 and 4:

$\circ_1$	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

$\circ_2$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

By Note 2.5.A, $T = \{\{a, b, a \circ_1 b\} \mid a < b \in Q\}$ so that for a 3-fold triple system, T contains the triples

$$\{1,2,5\},\{1,3,2\},\{1,4,6\},\{1,5,3\},\{1,6,7\},\{1,7,4\},\{2,3,6\},$$

$$\{2,4,3\},\{2,5,7\},\{2,6,4\},\{2,7,1\},\{3,4,7\},\{3,5,4\},\{3,6,1\},$$

$$\{3,7,5\},\{4,5,1\},\{4,6,5\},\{4,7,2\},\{5,6,2\},\{5,7,6\},\{6,7,3\}.$$

By Note 2.5.B, $T = \{\{a, b, a \circ_2 b\} \mid a, b \in Q, a \neq b\}$ so that for a 6-fold system, T contains the triples

$$\{1,2,3\},\{1,3,4\},\{1,4,2\},\{2,1,4\},\{2,3,1\},\{2,4,3\}$$

$$\{3,1,2\},\{3,2,4\},\{3,4,1\},\{4,1,3\},\{4,2,1\},\{4,3,2\}.$$