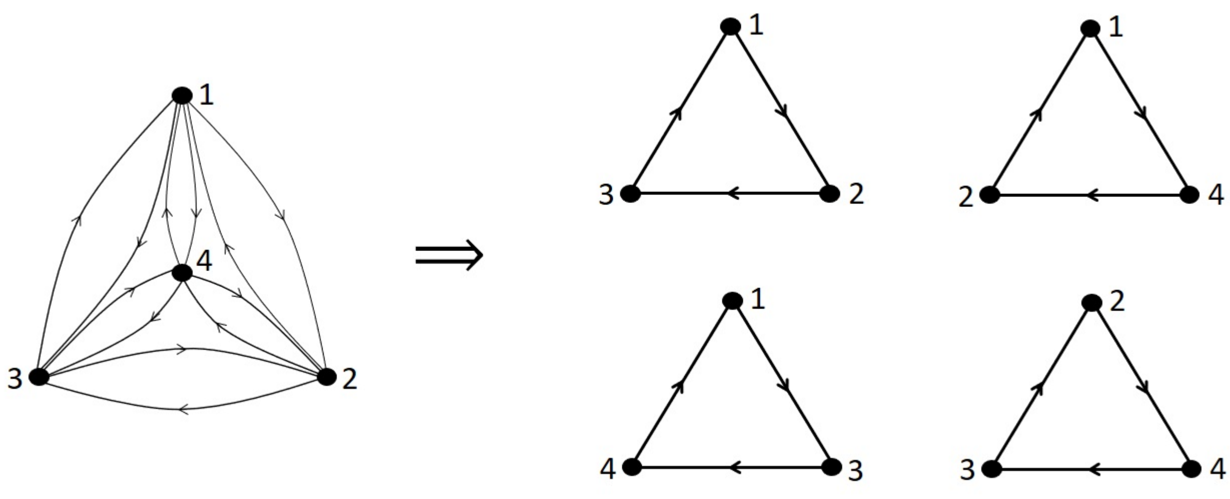
3.2. Mendelsohn Triple Systems Revisited

Note. In this section, we give two constructions. In one, we use a Mendelsohn triple system to construct an idempotent semisymmetric quasigroup; in the other, we use an idempotent semisymmetric quasigroup to construct a Mendelsohn triple system. This implies an equivalence of these two structures (of a given order).

Note. Recall from Section 2.4. Mendelsohn Triple Systems that a Mendelsohn triple system is a pair (Q, T) where T is an arc disjoint collection of directed triples which partitions the arc set of D_n on vertex set Q. (Remember the caution we take with the term "directed triple," as explained in Section 2.4.) In Example 2.4.1, we had the graphical representation of a Mendelsohn triple system of order 4 is as follows:



Note. We use the Mendelsohn triple system above to create a quasigroup. Define operation $\circ$ on $Q = \{1, 2, 3, 4\}$ by: (1) $a \circ a = a$ for all $a \in Q$, and (2) if $a \neq b$,

$a \circ b = c$ if and only if $(a, b, c) \in T$. We can easily check that the resulting quasigroup is:

$(Q, \circ) =$

$\circ$	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

By part (1) of the definition of the binary operation, we have that the identity $x \circ x = x$ holds. A quasigroup in which this holds is called "idempotent." We can check that the identity $x(yx) = y$ also holds, so that this quasigroup is semisymmetric (see Note 3.1.D).

Note. In fact, a Mendelsohn triple system of any order n can be used to construct an idempotent semisymmetric quasigroup of order n with the binary operation as described above. This is proved in Exercises 3.2.1 to 3.2.3. We then have the following construction of such quasigroups.

Note 3.2.A. We can construct an idempotent semisymmetric quasigroup $(Q, \circ)$, of order n from a Mendelsohn triple system of order n, (Q, T), by defining the binary operation $\circ$ on A as:

(1) $a \circ a = a$ for all $a \in Q$, and

(2) if $a \neq b$, $a \circ b = c$ if and only if $(a, b, c) \in T$.

Note 3.2.B. The construction of Note 3.2.A can be reversed so that an idempotent semisymmetric quasigroup of order n can be used to construct a Mendelsohn triple system. Let $(Q, \circ)$ be an idempotent semisymmetric quasigroup of order n and define a collection of directed triples T as: $(a, b, a \circ b) \in T$ for all $a \neq b \in T$. Then, as shown in Exercises 3.2.6 to 3.2.8, (Q, T) is a Mendelsohn triple system of order n. Since there is a construction in both directions, we see that a Mendelson triple system of order n is equivalent to an idempotent semisymmetric quasigroup of order n.

Definition. A quasigroup $(Q, \circ)$ which satisfies the identities $x \circ x = x$ (or "$x^2 = x$") and $x(yx) = y$ is a *Mendelsohn quasigroup*.

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