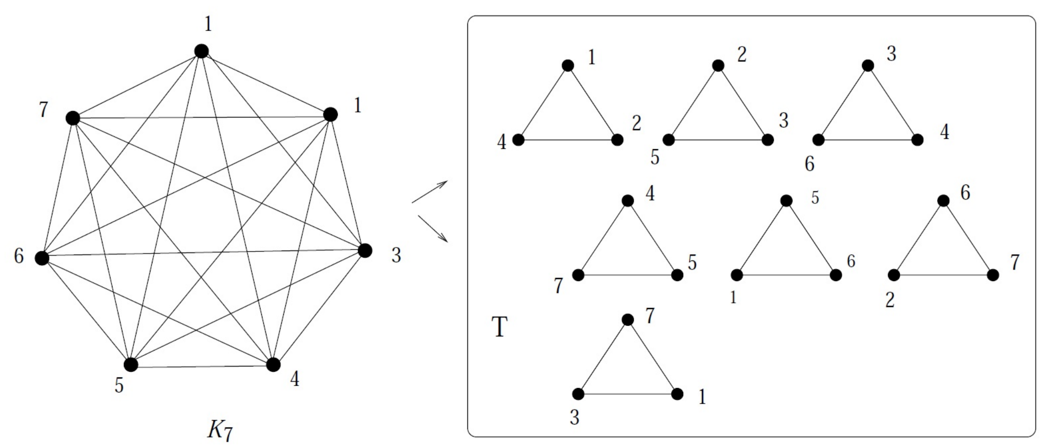
3.3. Steiner Triple Systems Revisited

Note. In this section, we give two constructions. In one, we use a Steiner triple system to construct a totally symmetric quasigroup; in the other, we use a totally symmetric quasigroup to construct a Steiner triple system. This implies an equivalence of these two structures (of a given order).

Note. Recall from Section 1.1. The Existence Problem that a Steiner triple system is a pair (S, T) where T is an edge disjoint collection of 3-cycles (or triples) which partitions the edge set of K_n on vertex set S. The graphical representation of a Steiner triple system of order 7 is as follows:



Notice that this is a cyclic $STS(7)$.

Note. We use the Steiner triple system above to create a quasigroup. Define operation $\circ$ on $Q = \{1, 2, 3, 4, 5, 6, 7\}$ by: (1) $a \circ a = a$ for all $a \in Q$, and (2) if $a \neq b$, $a \circ b = b \circ a = c$ if and only if $\{a, b, c\} \in T$. We can easily check that the resulting quasigroup is:

$(Q, \circ) =$

$\circ$	1	2	3	4	5	6	7
1	1	4	7	2	6	5	3
2	4	2	5	1	3	7	6
3	7	5	3	6	2	4	1
4	2	1	6	4	7	3	5
5	6	3	2	7	5	1	4
6	5	7	4	3	1	6	2
7	3	6	1	5	4	2	7

By part (1) of the definition of the binary operation, we have that the identity $x \circ x = x$ holds, so that this quasigroup is idempotent. By part (2) of the definition of the binary operation, we have the identity $xy = yx$ holds, so that this quasigroup if commutative. We can check that the identities $xy = yx$, $(yx)x = y$, $x(xy) = y$, $x(yx) = y$, and $(xy)x = y$ also holds, so that this quasigroup is totally symmetric.

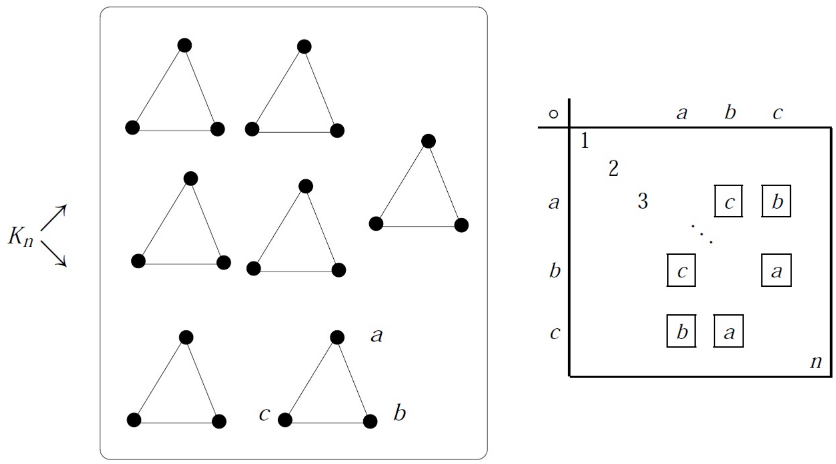
Note. In fact, a Steiner triple system of any order n can be used to construct an idempotent totally symmetric quasigroup of order n with the binary operation as described above. This is proved in Exercises 3.3.1 to 3.3.3. We then have the following construction of such quasigroups.

Note 3.3.A. We can construct an idempotent totally symmetric quasigroup $(Q, \circ)$, of order n from a Steiner triple system of order n, (S, T), by defining the binary operation $\circ$ on A as:

(1) $a \circ a = a$ for all $a \in Q$, and

(2) if $a \neq b$, $a \circ b = b \circ a = c$ if and only if $\{a, b, c\} \in T$.

Lindner and Rodger illustrate this construction as follows (and state the this is sometimes called the "Opposite Vertex Construction"):



Note 3.3.B. The construction of Note 3.3.A can be reversed so that an idempotent totally symmetric quasigroup of order n can be used to construct a Steiner triple system. Let $(Q, \circ)$ be an idempotent totally symmetric quasigroup of order n and define a collection of directed triples T as: $\{a, b, a \circ b\} \in T$ for all $a \neq b \in Q$. Then, as shown in Exercises 3.3.5 to 3.3.7, (Q, T) is a Steiner triple system of order n. Since there is a construction in both directions, we see that a Steiner triple system of order n is equivalent to an idempotent totally symmetric quasigroup of order n.

Definition. A quasigroup $(Q, \circ)$ which is equivalent to a Steiner triple system is a *Steiner quasigroup*.

Note 3.3.C. A Steiner quasigroup can also be defined in terms of certain identities. One set of such identities is $x^2 = x$, $xy = yx$, and $(yx)x = y$. Another possible set of such identities is $x^2 = x$, $xy = yx$, and $x(yx) = y$.

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