Chapter 4. Maximum Packings and Minimum Coverings

Note. In this chapter, we address a sort-of approximation theorem for Steiner triple systems. By Theorem 1.3.B, a Steiner triple system of order v exists if and only if $v \equiv 1$ or 3 (mod 6). We consider how close we can get to a Steiner triple system when v does not satisfy these conditions. We consider two approaches to "closeness," namely packings and coverings.

4.1. The General Problem

Note. In this section, we give two options for how to get "close" to a Steiner triple system when $v \equiv 0$, 2, 4, or 5 (mod 6). In these cases we know that there is not a collection of triples T which contains every pair of elements of set S exactly once. So we have two options: we can either *repeat* some pairs or we can *omit* some pairs in the collection of triples. We view these repeated pairs or omitted pairs as "bad," and want to minimize both. As with Steiner triple systems, there will be a graph theoretic interpretation of both approaches. In an attempt to minimize omitted pairs, we have the following.

Definition. A *packing* of the complete graph K_v with triangles is a triple (S, T, L), where S is the vertex set of K_v, T is a collection of edge disjoint triangles from the edge set of K_v, and L is the collection of edges in K_v not belonging to one of the triangles of T. The collection of edges L is the *leave*. If $|T|$ is as large as possible, or equivalently if $|L|$ is as small as possible, then (S, T, L) is a *maximum packing with triangles* (MPT), or simply a *maximum packing* of order v.

Note. A Steiner triple system is a maximum packing with leave $L = \varnothing$. We now give some additional examples.

Example 4.1.1. We consider examples of maximum packings with triples for $v \equiv 0 \pmod 6$, $v \equiv 4 \pmod 6$, and $v \equiv 5 \pmod 6$.

(a) A maximum packing with triples of order 6, (S_1, T_1, L_1) is given by:

$$\begin{aligned} S_1 &= \{1,2,3,4,5,6\}, \\ T_1 &= \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{1,5,6\}\}, \\ L_1 &= \{\{1,3\},\{2,6\},\{4,5\}\}. \end{aligned}$$

(b) A maximum packing with triples of order 10, (S_2, T_2, L_2) is given by:

$$\begin{aligned} S_2 &= \{1,2,3,4,5,6,7,8,9,10\}, \\ T_2 &= \{\{2,3,4\},\{1,6,7\},\{1,8,9\},\{1,5,10\},\{2,6,9\},\{2,5,7\},\{2,8,10\}, \\ &\quad \{3,5,6\},\{3,7,8\},\{3,9,10\},\{4,6,10\},\{4,7,9\},\{4,5,8\}\}, \\ L_2 &= \{\{1,2\},\{1,3\},\{1,4\},\{6,8\},\{7,10\},\{5,9\}\}. \end{aligned}$$

(c) A maximum packing with triples of order 11, (S_3, T_3, L_3) is given by:

$$\begin{aligned} S_3 &= \{1,2,3,4,5,6,7,8,9,10,11\}, \\ T_3 &= \{\{1,3,5\},\{2,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},\{2,6,9\},\{2,7,11\}, \\ &\quad \{2,8,10\},\{3,6,11\},\{3,7,8\},\{3,9,10\},\{4,6,10\},\{4,7,9\},\{4,8,11\}, \\ &\quad \{5,6,8\},\{5,7,10\},\{5,9,11\}\}, \\ L_3 &= \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}. \end{aligned}$$

Notice that the leave in (S_1, T_1, L_1), where $v = 6$, is a collection of three edges which contain each vertex exactly once. Such a subgraph is called a "1-factor."

More formally, a spanning subgraph H of a graph G is a 1-*factor* of G if H every vertex of H is of degree 1; see my online notes for Introduction to Graph Theory (MATH 4347/5347) on Section 2.2. Edge Colorings. A schematic of a maximum packing with triples of K_8 is given in Figure 4.1, where the leave is also a 1-factor (notice that $v = 6$ and $v = 8$ both fall in the category of values of $v \equiv 0$ or 2 (mod 6)).

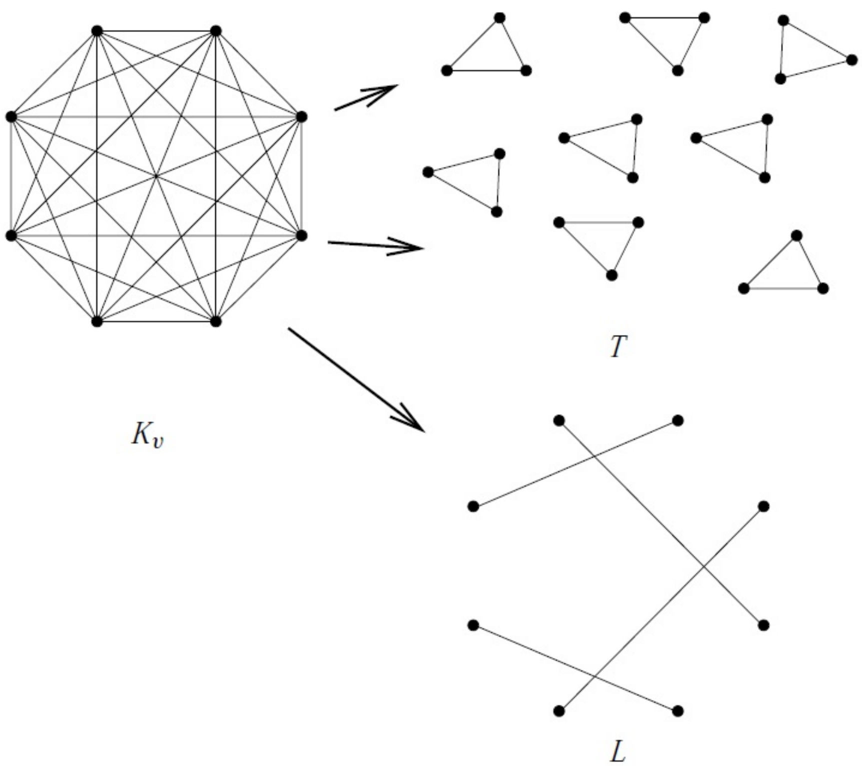


Figure 4.1: A maximum packing with leave being a 1-factor.

Note. We will see in the next section (see Theorem 4.2.A) that a maximal packing with triples has a leave of known structure (and size). As suggested in Example 4.1.1, we have that the leave L is:

(i) a 1-factor if $v \equiv 0$ or 2 (mod 6),

(ii) a 4-cycle if $v \equiv 5 \pmod 6$,

(iii) a *tripole*, that is a spanning graph with each vertex having odd degree and containing $(v+2)/2$ edges, if $v \equiv 4 \pmod 6$, and

(iv) the empty set if $v \equiv 1$ or $3 \pmod 6$.

See Figure 4.2.

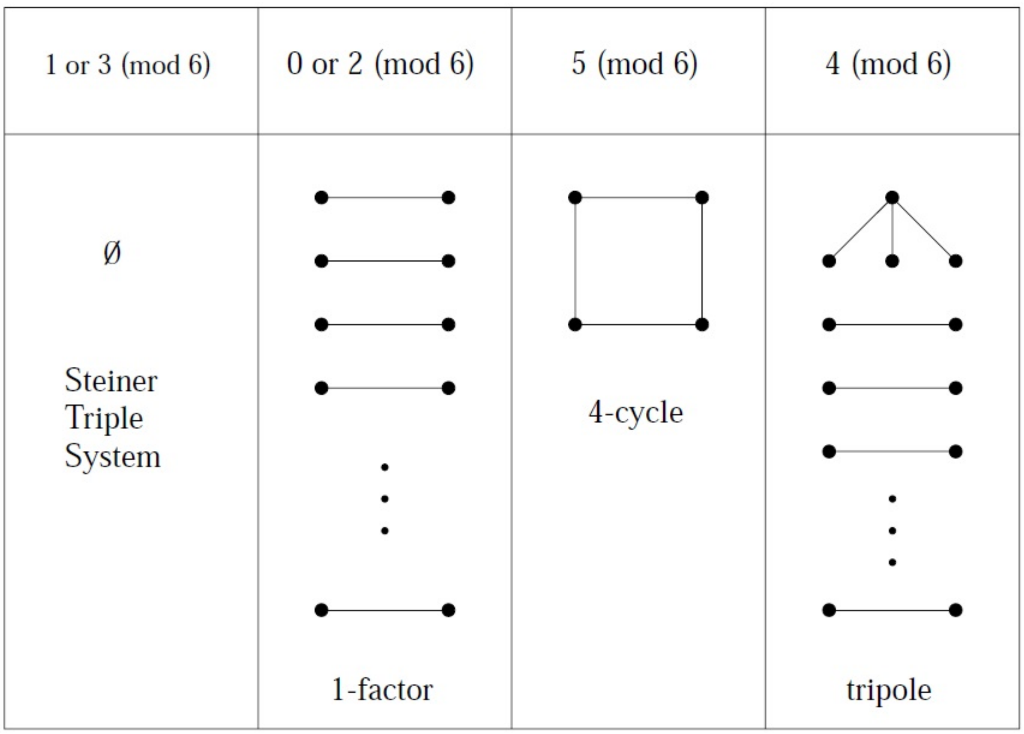


Figure 4.2: Leaves of maximum packings.

Note. In a packing of K_v, we minimized the ommitted pairs in the collection of triples. In our second idea approach to closeness to a Steiner triple system, we minimize the repeated pairs in the collection triples. This leads to the following definition.

Definition. A *covering* of the complete graph K_v with triangles is a triple (S, T, P), where S is the vertex set of K_v, P is a subset of the edge set of λK_v based on S (so P may have edges repeated multiples of times; technically we should speak of the "multiset" P and the edge multiset of λK_v), and T is a collection of edge disjoint triangles which partition the union of multiset P and the edge set of K_v. The collection of edges P is the *padding* and v is the *order* of the covering (S, T, P). If $|P|$ is as small as possible then the covering (S, T, P) is a *minimum covering with triangles* (MCT), or simply a *minimum covering* of order v.

Note. Notice that in an attempt to achieve "closeness" to Steiner triple systems, both maximal packings and minimal coverings are, in fact, minimization problems. This is to be expected, since every approximation is best when the distance from the desired quantity to the approximation is minimized. Here, we are measuring the "distance" between a packing or covering to a Steiner triple system by counting the number of omitted or repeated pairs, respectively.

Example 4.1.2. We consider examples of minimum coverings with triples for $v \equiv 5 \pmod 6$, $v \equiv 0 \pmod 6$, and $v \equiv 2 \pmod 6$.

(a) A minimum covering with triples of order 5, (S_1, T_1, P_1) (where P_1 may be a multiset) is given by:

$$\begin{aligned} S_1 &= \{1,2,3,4,5\}, \\ T_1 &= \{\{1,2,4\},\{1,2,3\},\{1,2,5\},\{3,4,5\}\}, \\ P_1 &= \{\{1,2\},\{1,2\}\}. \end{aligned}$$

(b) A minimum covering with triples of order 6, (S_2, T_2, P_2) (where P_2 may be a multiset) is given by:

$$
\begin{aligned}
S_2 &= \{1,2,3,4,5,6\},\\
T_2 &= \{\{1,2,3\},\{1,2,4\},\{3,4,5\},\{3,4,6\},\{2,5,6\},\{1,5,6\}\},\\
P_2 &= \{\{1,2\},\{3,4\},\{5,6\}\}.
\end{aligned}
$$

(c) A minimum covering with triples of order 8, (S_3, T_3, P_3) (where P_3 may be a multiset) is given by:

$$
\begin{aligned}
S_3 &= \{1,2,3,4,5,6,7,8\},\\
T_3 &= \{\{1,2,7\},\{1,4,5\},\{3,5,6\},\{1,2,3\},\{2,4,8\},\{5,7,8\},\\
&\quad \{1,3,8\},\{2,5,6\},\{6,7,8\},\{1,4,6\},\{3,4,7\}\},\\
P_3 &= \{\{1,2\},\{1,3\},\{1,4\},\{5,6\},\{7,8\}\}.
\end{aligned}
$$

Note. We will see in the Section 4.3 (see Theorem 4.3.A) that a minimal covering with triples has a leave of known structure (and size). As suggested in Example 4.1.2, we have that the padding P is:

(i) a 1-factor if $v \equiv 0 \pmod 6$,

(ii) a tripole if $v \equiv 2$ or $4 \pmod 6$,

(iii) a double edge, $\{\{a,b\},\{a,b\}\}$ if $v \equiv 5 \pmod 6$, and

(iv) the empty set if $v \equiv 1$ or $3 \pmod 6$.

See Figure 4.4.

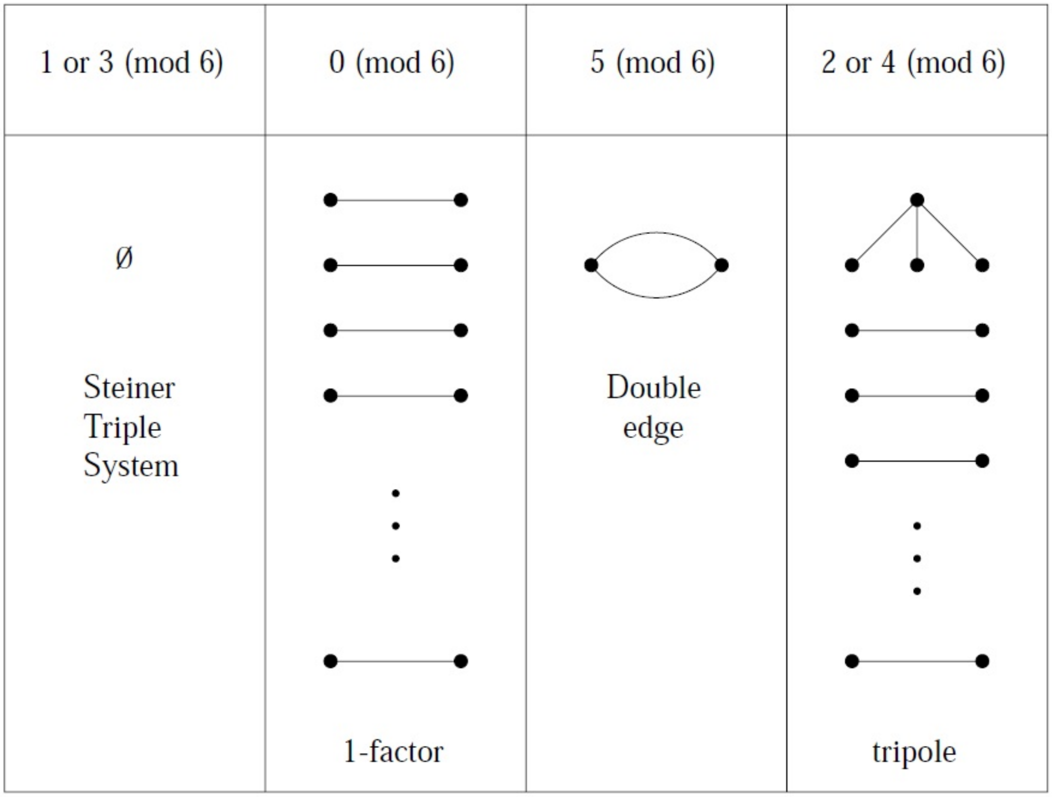
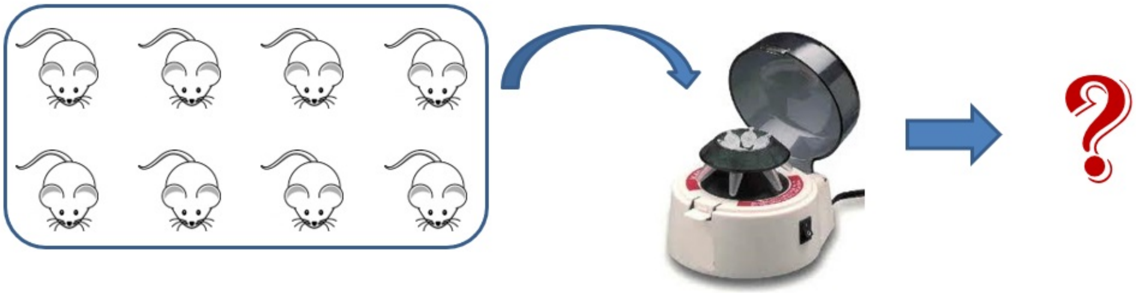


Figure 4.4: Paddings of minimum coverings.

Note. In the introduction to Chapter 1 (given in the online notes on Section 1.1. The Existence Problem), we motivated the study of Steiner triple systems by considering comparing some kind of samples in the fictional Motivationtron 3000. This story can be extending to maximal packings and minimal coverings. We can prioritize a maximal packing with a story about comparing as many samples as possible without repetition (because of expense in running the Motivationtron 3000, say). We can prioritize a minimal covering with a story about comparing *all* samples to each other, but still minimizing the number of runs of the Motivationtron 3000 (again because of expense, say). See my online talk “What the Hell do Graph Decompositions have to do with Experimental Designs,” prepared as a PowerPoint presentation. Notice in particular the sections on “What if a Graph Decomposition

Does Not Exist," and the subsections on "Graph Packing" and "Graph Covering."



Note. As a final comment, I want to draw a parallel between packings and coverings (as addressed in this section) and inner and outer measure (as dealt with in introductory measure theory). The outer measure of a set of real numbers is defined by covering the set with open intervals, adding up the lengths of the intervals, and then taking a minimum (well, an infimum) over all such coverings. This is similar to the covering of a complete graph with triples in such a way as to minimize the repeated edges (we can legitimately say "minimize" in the discrete setting). The inner measure of a set of real numbers is defined by packing compact sets inside the set, adding up their measures (defined in terms or outer measure), and then taking a maximum (well, a supremum) over all such packings. This is similar to the packing of a complete graph with triples in such a way as to minimize the missed edges. Outer measure is defined in Real Analysis 1 (MATH 5210); see my online notes on Section 2.2. Lebesgue Outer Measure. Inner measure is a supplemental topic in Real Analysis 1; see my supplemental online notes on An Alternate Approach to the Measure of a Set of Real Numbers. I have never seen this parallel between ideas from such different areas of math anywhere else. To me it seems a fairly natural analogy to draw. . .

Revised: 5/23/2022