4.2. Maximum Packings

Note. In this section, we give three constructions which allow us to give necessary and sufficient conditions for a maximal packing with triangles of K_v. The results of this section originally appear in: J. Schönheim, "On Maximal Systems of k-Tuples," *Studia Scientiarum Mathematicarum Hungarica*, **1**, 363–368 (1966). Historically, minimal coverings with triangles were studied first; we'll explore this in the next section. As is often the case, Lindner and Rodger present constructions different from (and simpler than) the original proofs.

Note. We claimed in Section 4.1 that a minimal packing with triples of K_v has a leave L of the following form:

(i) a 1-factor if $v \equiv 0$ or $2 \pmod 6$,

(ii) a 4-cycle if $v \equiv 5 \pmod 6$,

(iii) a *tripole*, that is a spanning graph with each vertex having odd degree and containing $(v+2)/2$ edges, if $v \equiv 4 \pmod 6$, and

(iv) the empty set if $v \equiv 1$ or $3 \pmod 6$.

See Figure 4.2 in Section 4.1. By Theorem 1.3.B, a Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$ and this covers case (iv). We now consider constructions in the other three cases.

Note 4.2.A. We first describe the construction for $v \equiv 0$ or $2 \pmod 6$, in which case the leave L is a 1-factor. Let (S, T) be a Steiner triple system of order $|S| = v+1 \equiv 1$

or 3 (mod 6). Let one of the elements of S be ∞ and let $T(\infty)$ be the set of all triples containing ∞ (so $|T(\infty)| = v/2$). Let $L = \{\{a,b\} \mid \{\infty, a, b\} \in T\}$. Then $(S \setminus \{\infty\}, T \setminus T(\infty), L)$ is a maximum packing of order v with leave the 1-factor L, as is to be shown in Exercises 4.2.4 and 4.2.5.

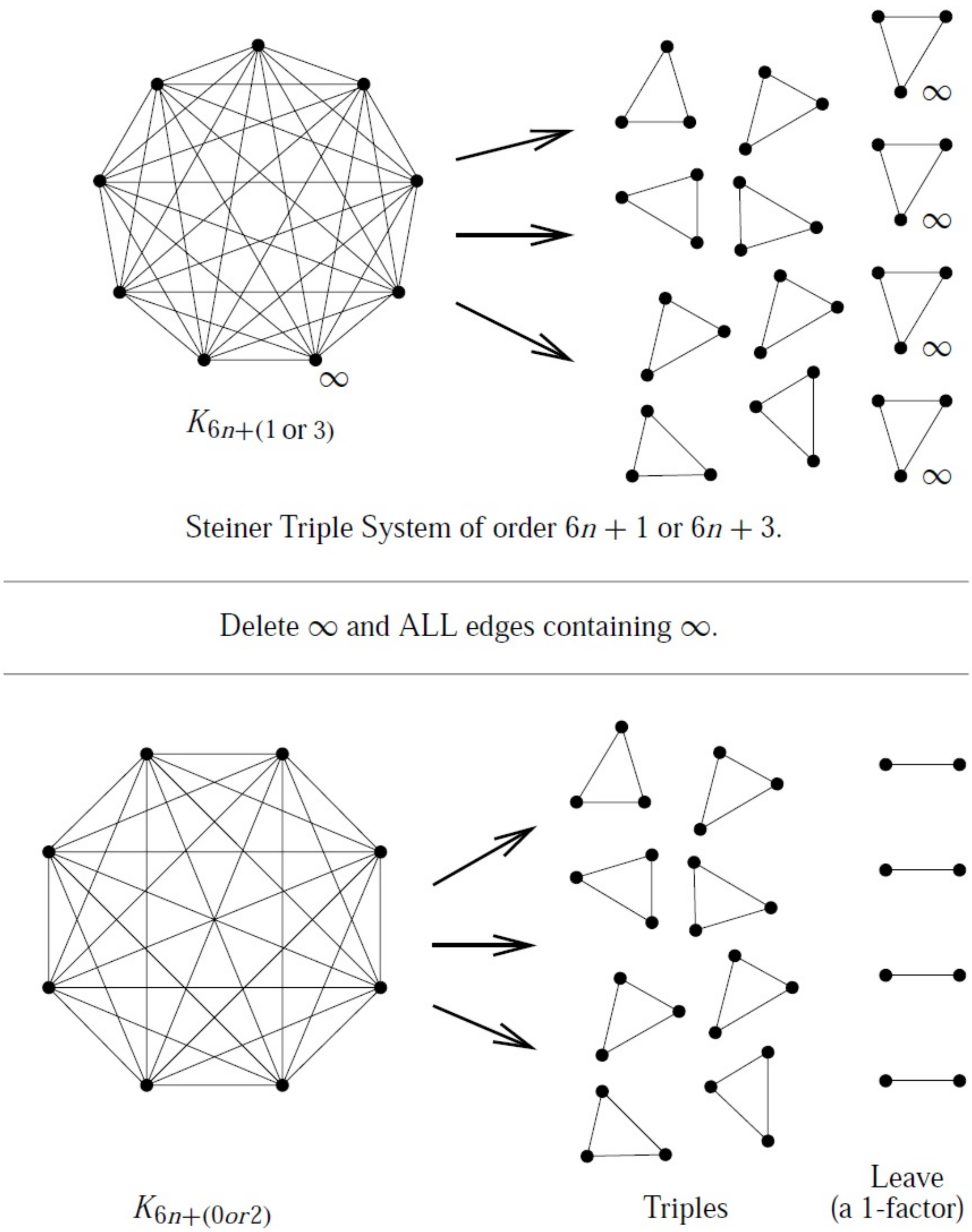

Figure 4.5: Maximum packing of order $6n + (0 \text{ or } 2)$ with leave a 1-factor.

Note 4.2.B. We now describe the construction for $v \equiv 5 \pmod 6$, in which case the leave L is a 4-cycle. Let (P, B) be a pairwise balanced design with one block of size 5 and the remaining blocks of size 3 (which exists by the construction given in Section 1.4. $v \equiv 5 \pmod 6$: The $6n + 5$ Construction). Let T be the set of triples in B. Replace the block $\{a, b, c, d, e\}$ of size 5 with the maximum packing $(\{a, b, c, d, e\}, T^*, L)$ where $T^* = \{\{a, b, c\}, \{a, d, e\}\}$ and $L = \{\{b, e\}, \{e, c\}, \{c, d\}, \{d, b\}\}$ (a 4-cycle). Then $(P, T \cup T^*, L)$ is a maximum packing of order v with leave the 4-cycle L, as is to be shown in Exercises 4.2.4 and 4.2.6.

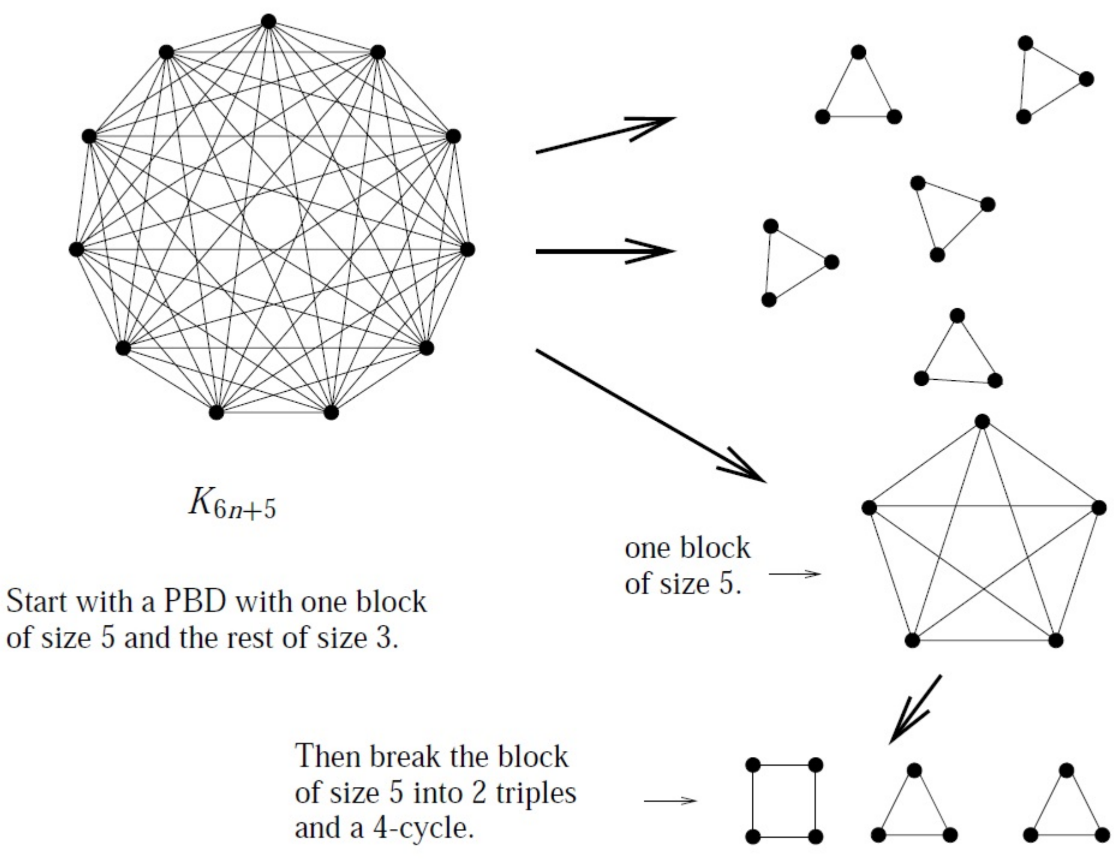

Figure 4.6: Maximum packing on order $6n + 5$ with leave a 4-cycle.

Note 4.2.C. We now describe the construction for $v \equiv 4 \pmod 6$, in which case the leave L is a tripole. Let (P, B) be a pairwise balanced design of order

$v+1$ with one block $\{\infty, a, b, c, d\}$ of size 5 and the remaining blocks of size 3 (which exists by the construction given in Section 1.4. $v \equiv 5 \pmod 6$: The $6n+5$ Construction). Let $B(\infty)$ be the set of blocks containing ∞, and let L be the set $\{\{x,y\} \mid \{x,y,\infty\} \in B\} \cup \{\{a,b\},\{a,c\},\{a,d\}$. Notice that L is a tripole. Then $(P \setminus \{\infty\}, B \setminus B(\infty)) \cup \{\{b,c,d\}\}, L)$ is a maximum packing of order v with leave the L is a tripole, as is to be shown in Exercises 4.2.4 and 4.2.7.

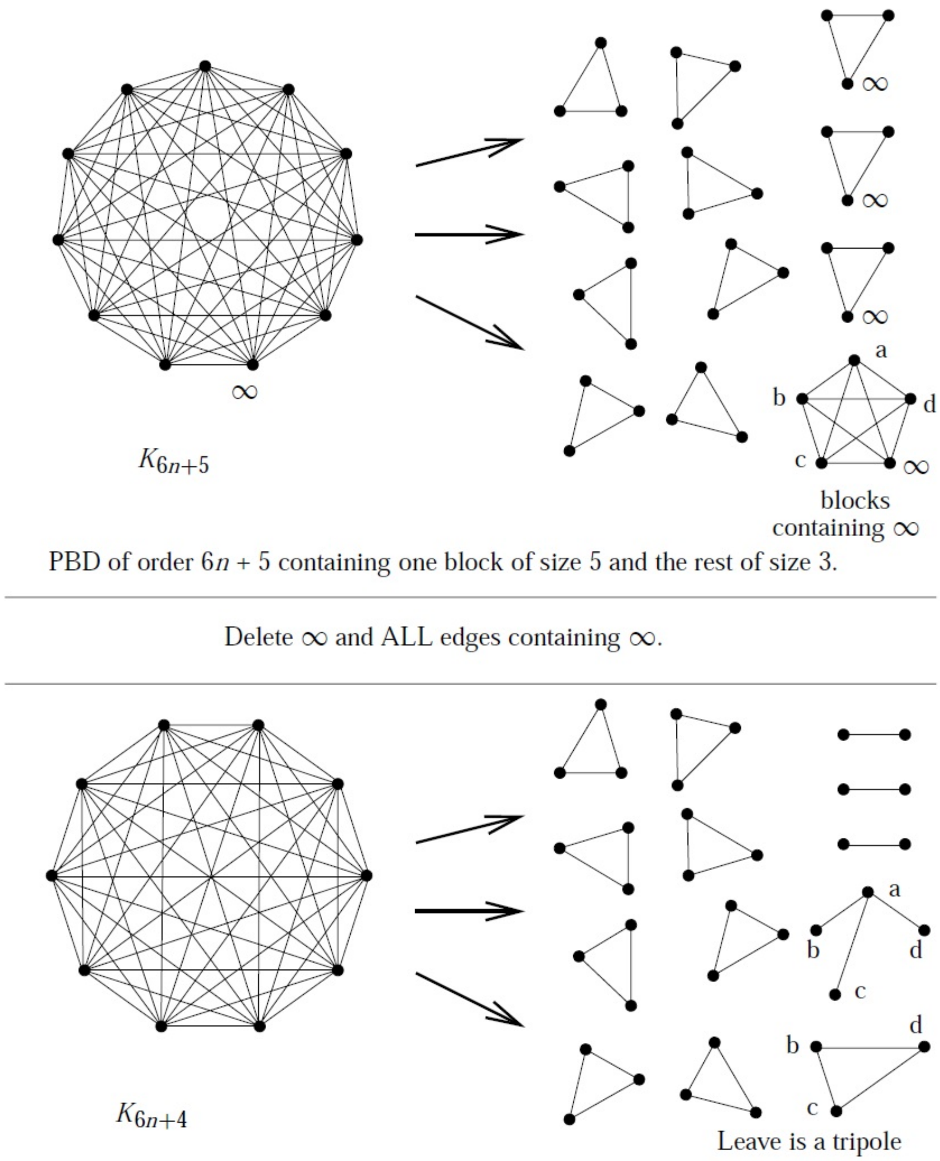

Figure 4.7: Maximum packing of order $6n+4$ with leave a tripole.

Note. Combining the results of Notes 4.2.A, 4.2.B, 4.2.C and Exercises 4.2.4, 4.2.5, 4.2.6, 4.2.7, we have the following classification of minimal packings with triples of K_v.

Theorem 4.2.A. A maximal packing with triples of K_v with leave L satisfies:

(i) if $v \equiv 0$ or $2 \pmod 6$ then L is a 1-factor with $v/2$ edges,

(ii) if $v \equiv 5 \pmod 6$ then L is a 4-cycle,

(iii) if $v \equiv 4 \pmod 6$ then L is aa *tripole* (that is a spanning graph with each vertex having odd degree and containing $(v+2)/2$ edges), and

(iv) if $v \equiv 1$ or $3 \pmod 6$, the $L = \varnothing$.

Note. We now illustrate the three constructions with three examples. These help illustrate how various symbols and triples are omitted in the final creation of the desired packing.

Example 4.2.1. For a maximum packing with triples of order 6, consider:

$$\begin{aligned} S &= \{\infty, 1, 2, 3, 4, 5, 6\}, \\ T &= \{\{\infty, 1, 3\}, \{\infty, 4, 5\}, \{\infty, 2, 6\}, \{1, 2, 4\}, \{2, 3, 5\}, \{1, 5, 6\}, \{3, 4, 6\}\}, \end{aligned}$$

The MPT of order 6 is then given by $(S \setminus \{\infty\}, T \setminus T(\infty), L)$, where L is the 1-factor $L = \{\{1, 3\}, \{4, 5\}, \{2, 6\}\}$.

Example 4.2.2. For a maximum packing with triples of order 11, consider the pairwise balanced design (P, B) or order 11:

$$\begin{aligned} P &= \{1,2,3,4,5,6,7,8,9,10,11\}, \\ B &= \{\{1,3,5,8,11\},\{1,2,9\},\{1,4,7\},\{2,8,10\},\{3,4,9\},\{1,6,10\}, \\ &\quad \{3,7,10\},\{5,6,9\},\{2,3,6\},\{4,5,10\},\{7,8,9\},\{2,4,11\},\{4,6,8\}, \\ &\quad \{9,10,11\},\{2,5,7\},\{6,7,11\}\}, \end{aligned}$$

Notice that a PBD of order 11 is given in Example 1.4.1(b), but this one is different. The MPT of order 11 is then given by (P, T, L), where $T = (B \setminus \{1,3,5,8,11\}) \cup \{\{1,3,5\},\{1,8,11\}\}$, and the leave L is the 4-cycle $L = \{\{3,8\},\{8,5\},\{5,11\},\{11,3\}\}$.

Example 4.2.3. For a maximum packing with triples of order 10, consider the pairwise balanced design (P, B) or order 11:

$$\begin{aligned} P &= \{\infty,1,2,3,4,5,6,7,8,9,10\}, \\ B &= \{\{1,3,5,8,\infty\},\{1,2,9\},\{1,4,7\},\{2,8,10\},\{3,4,9\},\{1,6,10\}, \\ &\quad \{3,7,10\},\{5,6,9\},\{2,3,6\},\{4,5,10\},\{7,8,9\},\{2,4,\infty\},\{4,6,8\}, \\ &\quad \{9,10,\infty\},\{2,5,7\},\{6,7,\infty\}\}. \end{aligned}$$

This is the same PBD or order 11 used in Example 4.2.2, except that the symbol "11" there is replaced with the symbol "∞" here. The MPT of order 10 is then given by $(P \setminus \{\infty\}, (B \setminus B(\infty)) \cup \{\{1,3,5\}\}, L)$, where L is the tripole $L = \{\{8,1\},\{8,3\},\{8,5\},\{2,4\},\{6,7\},\{9,10\}\}$.