4.3. Minimum Coverings

Note. In this section, we give three constructions which allow us to give necessary and sufficient conditions for a minimal covering with triangles of K_v. The results of this section originally appear in: Marion K. Fort, Jr. and G. A. Hedlund, "Minimal Coverings of Pairs by Triples," *Pacific Journal of Mathematics*, **8**(4), 709–719 (1958). A copy is online on the Mathematical Sciences Publishers webpage (accessed 5/23/2022). Lindner and Rodger present constructions different from (and simpler than) the original proofs.

Note. We claimed in Section 4.1 that a minimal covering with triple of K_v has a padding P of the following form:

(i) a 1-factor if $v \equiv 0 \pmod{6}$,

(ii) a tripole if $v \equiv 2$ or $4 \pmod{6}$,

(iii) a double edge, $\{\{a,b\},\{a,b\}\}$ if $v \equiv 5 \pmod{6}$, and

(iv) the empty set if $v \equiv 1$ or $3 \pmod{6}$.

See Figure 4.4 in Section 4.1. By Theorem 1.3.B, a Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod{6}$ and this covers case (iv). We now consider constructions in the other three cases.

Note 4.3.A. We first describe the construction for $v \equiv 0 \pmod{6}$, in which case the padding P is a 1-factor. Let (X, B) be a pairwise balanced design of order $v - 1 \equiv 5 \pmod{6}$ with one block $\{a, b, c, d, e\}$ of size 5 and the remaining blocks of size 3 (which exists by Section 1.4. $v \equiv 5 \pmod{6}$: The $6n + 5$ Construction). Denote by T the collection of blocks of size 3. Let $S = \{\infty\} \cup X$ and let $\pi = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_t, y_t\}\}$ be any partition of $X \setminus \{a, b, c, d, e\}$ (so that $t = (v - 6)/2$). Define $\pi(\infty) = \{\{\infty, x_1, y_1\}, \{\infty, x_2, y_2\}, \ldots, \{\infty, x_t, y_t\}$ and $F(\infty) = \{\{\infty, a, e\}, \{\infty, b, e\}, \{\infty, c, e\}, \{c, d, e\}, \{a, b, d\}, \{a, b, c\}\}$. (Notice that $F(\infty)$ is a minimal covering with triples of order 6.) Then (S, T^*, P) is a minimum covering of order v where $T^* = T \cup \pi(\infty) \cup F(\infty)$, and the the padding P is $P = \pi \cup \{\{a, b\}, \{c, d\}, \{e, \infty\}\}$, as is to be shown in Exercises 4.3.4 and 4.3.5. See Figure 4.8 below.

Note 4.2.B. We now describe the construction for $v \equiv 5 \pmod{6}$, in which case the padding P is a double edge. Let (S, B) be a pairwise balanced design with one block of size 5, $\{a, b, c, d, e\}$, and the remaining blocks of size 3 (which exists by Section 1.4. *vequiv*5 (mod 6): The $6n + 5$ Construction). Denote by T the collection of blocks of size 3, and let $T^* = \{\{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{c, d, e\}\}$. Then $(S, T \cup T^*, P)$ is a minimum covering of order v, where $P = \{\{a, b\}, \{a, b\}\}$, as is to be shown in Exercises 4.3.4 and 4.3.6. See Figure 4.9 below.

Note 4.2.C. We now describe the construction for $v \equiv 2$ or $4 \pmod 6$, in which case the padding P is a tripole. Let (X, T) be a Steiner triple system of order $v - 1 \equiv 1$ or $3 \pmod 6$ (which exists by Theorem 1.3.B). Let $\{a, b, c\} \in T$ and let $\pi = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_t, y_t\}\}$ be any partition of $X \setminus \{\{a, b, c\}\}$. Let $\pi(\infty) = \{\{\infty, x_1, y_1\}, \{\infty, x_2, y_2\}, \ldots, \{\infty, x_t, y_t\}\}$ and $(\infty) = \{\{\infty, a, b\}, \{\infty, b, c\}, \{a, b, c\}\}$. Let $S = \{\infty\} \cup X$. Then (S, T^*, P) is a minimum covering of order v, where $T^* = T \cup \pi(\infty) \cup T(\infty)$, and the padding P is the triple $P = \pi\{\{a, b\}, \{\infty, b\}, \{b, c\}\}$, as is to be shown in Exercises 4.3.4 and 4.3.7. See Figure 4.10 below.

Note. Combining the results of Notes 4.3.A, 4.3.B, 4.3.C and Exercises 4.3.4, 4.3.5, 4.3.6, 4.3.7, we have the following classification of minimal coverings with triples of K_v.

Theorem 4.3.A. A minimal covering with triples of K_v has a padding P of the following form:

(i) a 1-factor if $v \equiv 0 \pmod 6$,

(ii) a tripole if $v \equiv 2$ or $4 \pmod 6$,

(iii) a double edge, $\{\{a, b\}, \{a, b\}\}$ if $v \equiv 5 \pmod 6$, and

(iv) the empty set if $v \equiv 1$ or $3 \pmod 6$.

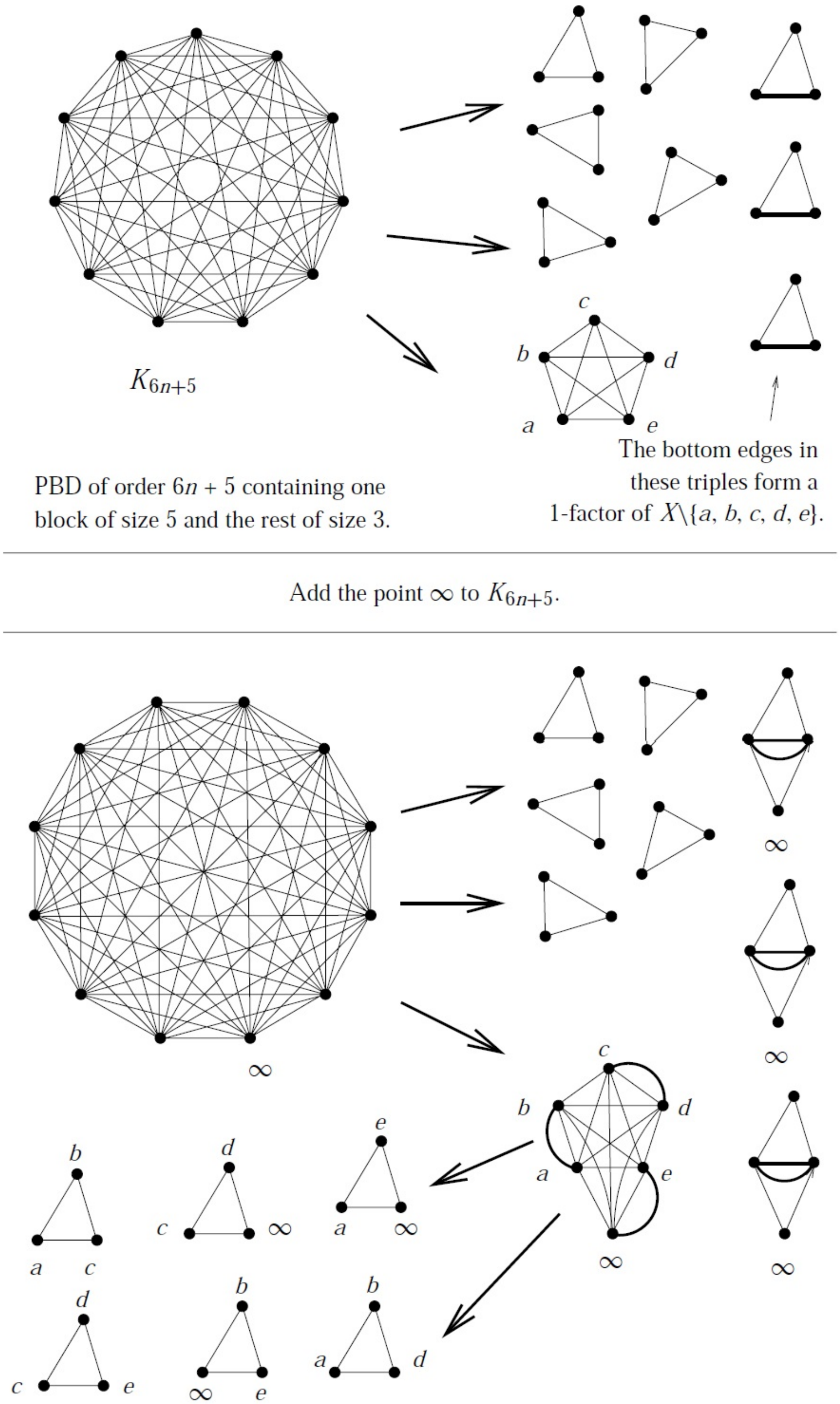
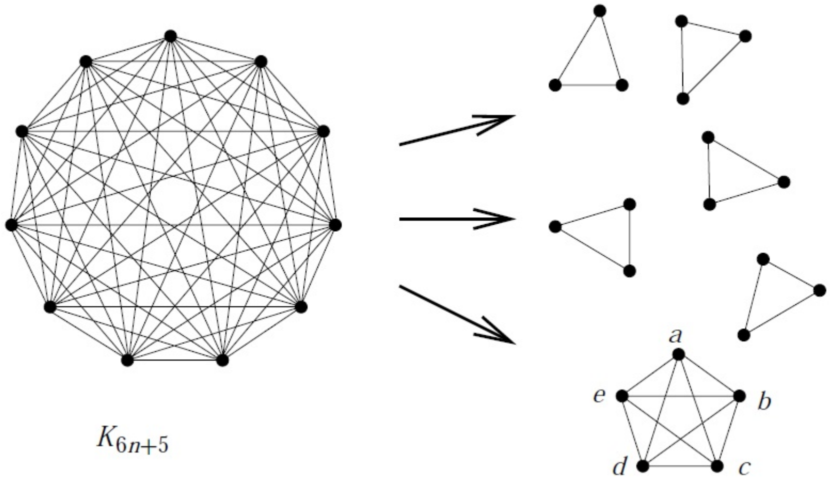


Figure 4.8: Minimum covering of order $6n$ with padding a 1-factor.



PBD of order $6n + 5$ containing one block of size 5 and the rest of size 3.

Add the double edge $\{a, b\}$, $\{a, b\}$ to K_{6n+5}.

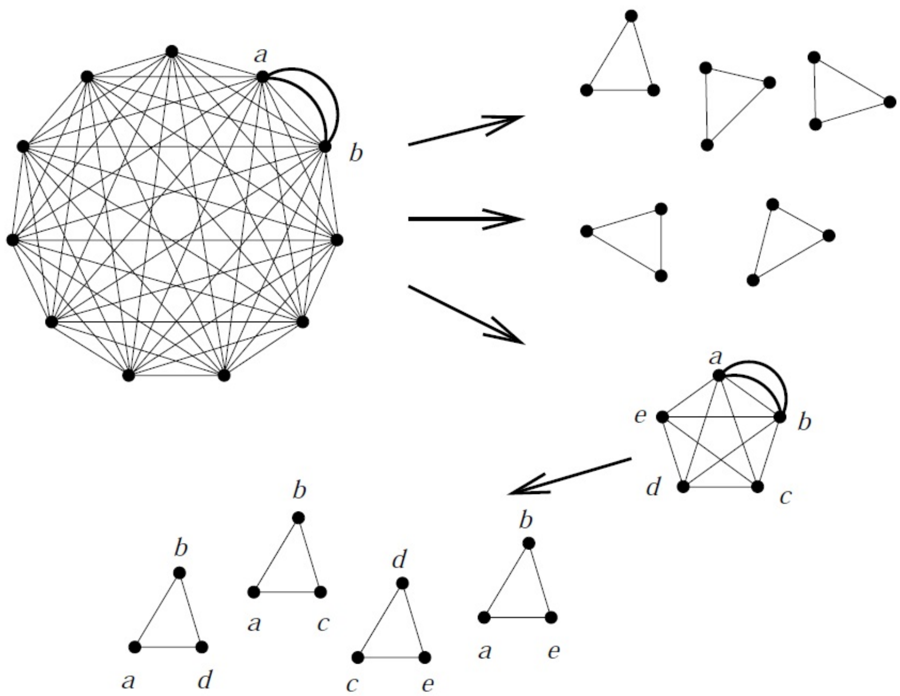


Figure 4.9: Minimum covering of order $6n + 5$ in which the padding is a double edge.

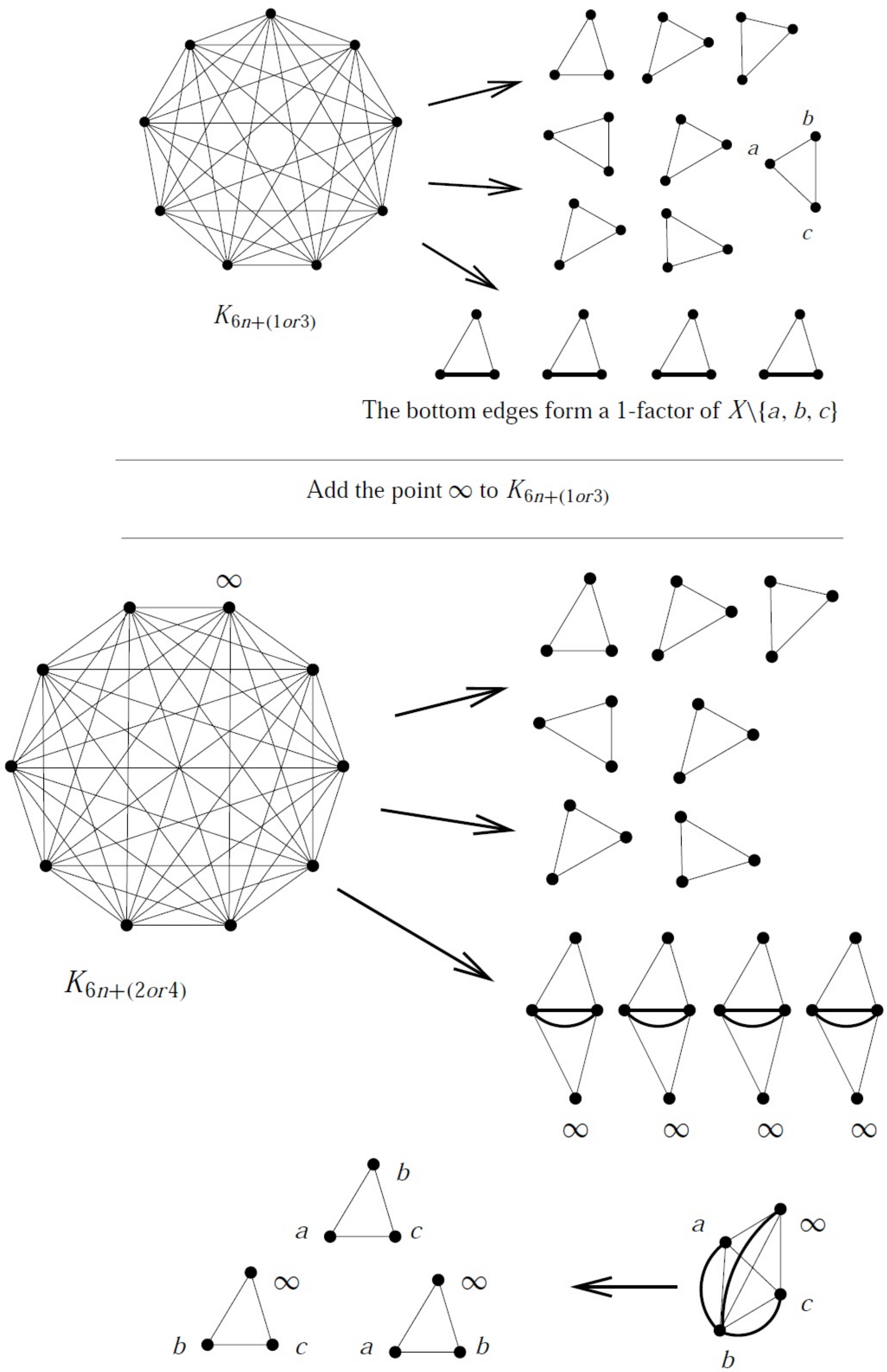


Figure 4.10: Minimum covering of order $v \equiv 2$ or 4 (mod 6) with padding a tripole.

Note. We now illustrate the three constructions with three examples. These help illustrate how various symbols and triples are omitted in the final creation of the desired packing.

Example 4.3.1. For a minimal covering with triples of order 12, consider:

$$\begin{aligned} X &= \{1,2,3,4,5,6,7,8,9,10,11\}, \\ B &= \{\{1,3,5,8,11\},\{1,2,9\},\{1,4,7\},\{2,8,10\},\{3,4,9\},\{1,6,10\}, \\ &\quad \{3,7,10\},\{5,6,9\},\{2,3,6\},\{4,5,10\},\{7,8,9\},\{2,4,11\},\{4,6,8\}, \\ &\quad \{9,10,11\},\{2,5,7\}\{6,7,11\}\} \\ F(\infty) &= \{\{\infty,1,11\},\{\infty,3,11\},\{\infty,5,8\},\{3,5,8\}, \\ &\quad \{1,3,8\},\{1,3,5\}\} \end{aligned}$$

Let $\pi = \{\{2,4\},\{6,7\},\{9,10\}\}$ and $S = \{\infty\} \cup X$. Then (S, T^*, P) is a minimum covering of order 12 with padding the 1-factor $P = \pi \cup \{1,3\},\{5,8\},\{\infty,11\}\}$.

Example 4.3.2. For a minimum covering with triples of order 11, we consider again the minimum covering of order 12 from Example 4.3.1, (S, T^*, P) with padding the 1-factor $P = \pi \cup \{1,3\},\{5,8\},\{\infty,11\}\}$. Replace the block $\{1,3,5,8,11\}$ with the triples $T^* = \{\{1,3,5\},\{1,3,8\},\{1,3,11\},\{5,8,11\}\}$. Then $(X, B \setminus \{\{1,3,5,8,11\}\} \cup T^*, P)$ is a minimum covering of order 11 with padding P which is the double edge $P = \{\{1,3\},\{3,1\}\}$. Here, $X = \{1,2,\ldots,11\}$ so that it is the symbol ∞ that is not present here.

Example 4.3.3. For a minimal covering with triples of order 8, consider a Steiner triple system of order 7, (X, T), where $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $T = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$ (this is the cyclic $STS(7)$ of Note 1.1.A). Let $\pi = \{\{3, 5\}, \{6, 7\}\}$ be a partition of $X \setminus \{\{1, 2, 4\}\}$, $\pi(\infty) = \{\{\infty, 3, 5\}, \{\infty, 6, 7\}\}$, and $T(\infty) = \{\{\infty, 1, 2\}, \{\infty, 2, 4\}, \{1, 2, 4\}\}$. Then (S, T^*, P) is a minimum covering of order 8, where $T^* = T \cup \pi(\infty) \cup T(\infty)$, and the padding P is the tripole $\{\{3, 5\}, \{6, 7\}, \{2, 1\}, \{2, 4\}, \{2, \infty\}\}$.

Revised: 11/30/2022