Chapter 7. Affine and Projective Planes

7.1. Affine Planes

Note. In this section, we define affine planes in terms of pairwise balanced designs with certain properties. We claim an equivalence between affine planes of order n and PBD on n^2 points where each block contains n points (in Exercises 7.1.3 and 7.1.4).

Note. Recall (from Section 1.4. $v \equiv 5 \pmod 6$: The $6n + 5$ Construction) that a pairwise balanced design, or PBD, is an ordered pair (S, B), where S is a finite set of symbols, and B is a collection of subsets of S called blocks, such that each pair of distinct elements of S occurs together in exactly one block of B. The order of the PBD is $|S|$.

Note/Definition. In this chapter, we call the blocks of a pairwise balanced design *lines*. Points which belong to the same line are *collinear*. Two lines that do not intersect (that is, do not share a point) are *parallel*. If point p belongs to line ℓ then p is *on the line* and ℓ *passes through* p.

Definition. An *affine plane* is a PBD (P, B) with the following properties:

(1) P contains at least one subset of 4 points, no 3 of which are collinear, and

(2) given a line ℓ and a point p not on ℓ, there is exactly one line of B containing p which is parallel to ℓ

The first property guarantees that the plane does not consist of a line. The second property is called the *Parallel Postulate*. Since, in a PBD, each pair of distinct elements of S occurs together in exactly one block of B then two points of an affine plane determine a unique line.

Example 7.1.1. **(a)** Consider $P = \{1,2,3,4\}$ and $B = \{\{1,2\}, \{1,3\}, \{1,4\}, \{3,4\}, \{2,4\}, \{2,3\}\}$. This is an affine plane on 4 points.
(b) Consider $P = \{1,2,3,4,5,6,7,8,9\}$ and

$$\begin{aligned} B = \{\{1,2,3\},\ & \{1,4,7\},\ \{1,5,9\},\ \{1,6,8\}, \\ \{4,5,6\},\ & \{2,5,8\},\ \{2,6,7\},\ \{2,4,9\}, \\ \{7,8,9\},\ & \{3,6,9\},\ \{3,4,8\},\ \{3,5,7\}\}. \end{aligned}$$

This is an affine plane on 9 points. Notice that 4 points, no 3 of which are collinear, are points 1, 2, 6, and 9 (for example). The Parallel Postulate is suggested by the presentation of B in columns.

Note. In Exercise 7.1.3, an affine plane (P, B) is considered. It is to be shown that for $n \geq 2$, the following are equivalent:

(a) One line contains n points.

(b) One point belongs to exactly $n + 1$ lines.

(c) Every line contains n points.

(d) Every point is on exactly $n + 1$ lines.

(e) There are exactly n^2 points in P.

(f) There are exactly $n^2 + n$ lines in B.

Definition. The number n of Exercise 7.1.3 is the *order* of the affine plane (P, B) (i.e., the order is the number of points on each line).

Note. Exercise 7.1.3 shows that an affine plane is a PBD on n^2 points where each block contains n points. In Exercise 7.1.4 the converse is to be shown: A PBD of order n^2 with block size n is an affine plane. Notice that the term "order" plays different roles in the setting of a PBD (where it represents the number of points) and in an affine plane (where it represents the number of points on a line).

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