7.2. Projective Planes

Note. In this section, we define projective planes in terms of pairwise balanced designs with certain properties. We claim an equivalence between projective planes on n^2+n+1 where each line contains $n+1$ points is equivalent to a PBD of order n^2+n+1 with block size $n+1$ (in Exercises 7.2.2 and 7.2.3).

Definition. A *projective plane* is a PBD (P, B) with the following properties:

(1) P contains at least one subset of 4 points, no 3 of which are collinear, and

(2) every pair of lines intersect in exactly one point.

The first property guarantees that the plane does not consist of a line. The second property implies that there are no parallel lines in a projective plane. Since, in a PBD, each pair of distinct elements of S occurs together in exactly one block of B then two points of a projective plane determine a unique line.

Note. Without the first property, we would admit the *degenerate projective plane* which consists of $n+1$ points with one line containing c of the points and the rest of the lines containing 2 points each. See Figure 7.3.

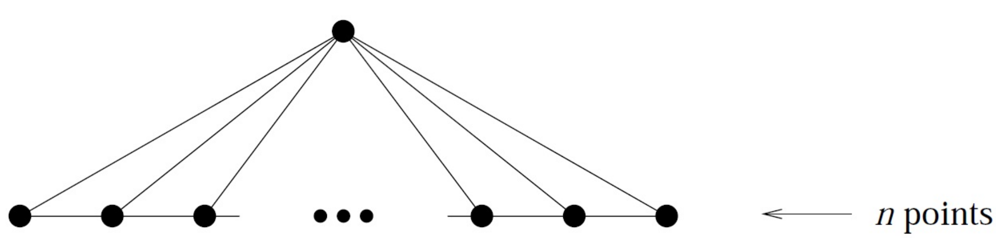

Figure 7.3: Degenerate projective plane (*not* a projective plane according to the definition).

Note. In my online notes for Introduction to Modern Geometry (MATH 4157/5157), a finite projective plane is axiomatically defined in Section 1.7. Finite Geometries. Other notes used in this class covering infinite projective spaces are online for Section 60. The Complex Projective Plane and Section 61. A Model for the Projective Plane.

Example 7.2.1. (a) Consider $P^* = \{1, 2, 3, 4, 5, 6, 7\}$ and $B^* = \{\{1, 2, 5\}, \{1, 3, 6\}, \{1, 4, 7\}, \{5, 6, 7\}, \{3, 4, 5\}, \{2, 4, 6\}, \{2, 3, 7\}\}$. This is a projective plane on 7 points. Notice that no three of the four points 1, 3, 5, 7 are collinear, as required.

Note. In Exercise 7.2.2, a projective plane (P, B) is considered. It is to be shown that for $n \geq 2$, the following are equivalent:

(a) One line contains $n + 1$ points.

(b) One point belongs to exactly $n + 1$ lines.

(c) Every line contains $n + 1$ points.

(d) Every point is on exactly $n + 1$ lines.

(e) There are exactly $n^2 + n + 1$ points in P.

(f) There are exactly $n^2 + n + 1$ lines in B.

Definition. The number n of Exercise 7.2.2 is the *order* of the projective plane plane (P, B) (i.e., the order is the number of points on each line minus 1).

Note. Exercise 7.2.2 shows that a projective plane is a PBD on $n^2 + n + 1$ points where each line contains $n+1$ points. In Exercise 7.2.3 the converse is to be shown: A PBD of order $n^2 + n + 1$ with block size $n + 1$ is a projective plane. Notice that the term "order" plays different roles in the setting of a PBD (where it represents the number of points) and in a projective plane (where it represents one less than the number of points on a line).

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