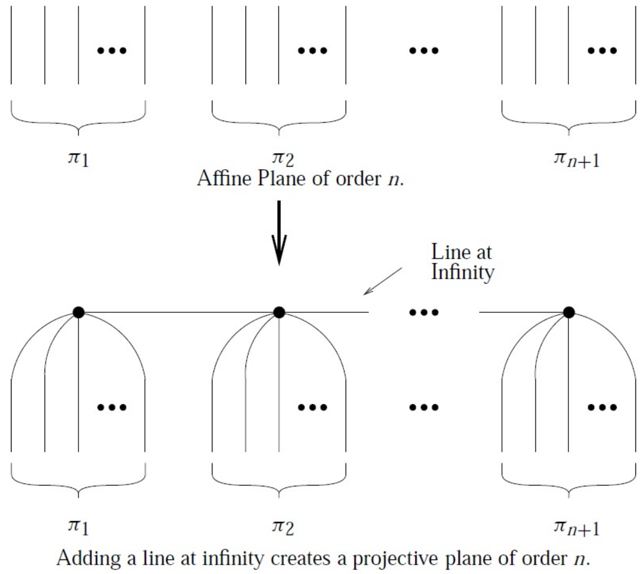
7.3. Connections between Affine and Projective Planes

Note. In this section, we state exercises that allow us to create an affine plane of order n from a projective plane of order n (Exercise 7.3.3), and conversely to create a projective plane of order n from an affine plane of order n (Exercise 7.3.5).

Definition. In an affine plane (P, B) a collection of mutually parallel lines which partition the points of P is a *parallel class* (also sometimes called a *pencil* of lines).

Note. In Exercise 7.3.2 it is to be shown that an affine plane of order n (that is, every line contains n points) has exactly $n + 1$ parallel classes, each containing n lines.

Note. In Exercise 7.3.3, it is shown that an affine plane of order n (that is, every line contains n points) can be used to create a projective plane of order n (that is, every line contains $n+1$ points). This is done by starting with affine plane of order n (P, B) and considering the $n+1$ parallel classes which are known to exist by Exercise 7.3.2. Say the parallel classes are the sets of lines $\pi_1, \pi_2, \ldots, \pi_{n+1}$. Consider the new (distinct) points $\infty_1, \infty_2, \ldots, \infty_{n+1}$ and the new line $\infty = \{\infty_1, \infty_2, \ldots, \infty_{n+1}\}$. Set $P(\infty) = P \cup \{\infty_1, \infty_2, \ldots, \infty_{n+1}\}$ and $B(\infty) = \{b \cup \infty_i\} \mid b \in \pi_i\} \cup \{\infty\}$. It is then to be proved that $(P(\infty), B(\infty))$ is a projective plane of order n. This technique is called *adding a line at infinity* and ∞ is the *line at infinity*. A diagram of the construction is:



Example. (This is Example 7.2.1(b).) To illustrate the construction above, we consider the affine plane from Example 7.1.1(b): $P = \{1,2,3,4,5,6,7,8,9\}$ and

$$\begin{aligned} B = \{&\{1,2,3\}, \ \{1,4,7\}, \ \{1,5,9\}, \ \{1,6,8\}, \\ &\{4,5,6\}, \ \{2,5,8\}, \ \{2,6,7\}, \ \{2,4,9\}, \\ &\{7,8,9\}, \ \{3,6,9\}, \ \{3,4,8\}, \ \{3,5,7\}\}. \end{aligned}$$

The four parallel classes are given as the columns here. Denote these columns as $\pi_1, \pi_2, \pi_3, \pi_4$. Introduce the new line $\infty = \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Then consider $P^* = \{1,2,3,4,5,6,7,8,9,\infty_1,\infty_2,\infty_3,\infty_4\}$ and

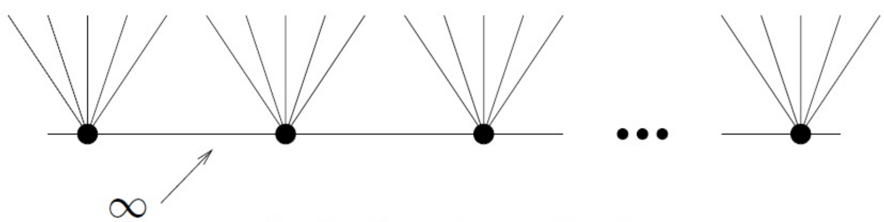
$$\begin{aligned} B^* = \{&\{1,2,3,\infty_1\}, \ \{1,4,7,\infty_2\}, \ \{1,5,9,\infty_3\}, \ \{1,6,8,\infty_4\}, \\ &\{4,5,6,\infty_1\}, \ \{2,5,8,\infty_2\}, \ \{2,6,7\infty_3\}, \ \{2,4,9,\infty_4\}, \\ &\{7,8,9,\infty_1\}, \ \{3,6,9,\infty_2\}, \ \{3,4,8\infty_3\}, \ \{3,5,7,\infty_3\}, \{\infty_1,\infty_2,\infty_3,\infty_4\}. \end{aligned}$$

This gives exactly the projective plane of Example 7.2.1(b) by replacing the symbols $\infty_1 \to 10$, $\infty_2 \to 11$, $\infty_3 \to 12$, and $\infty_4 \to 13$.

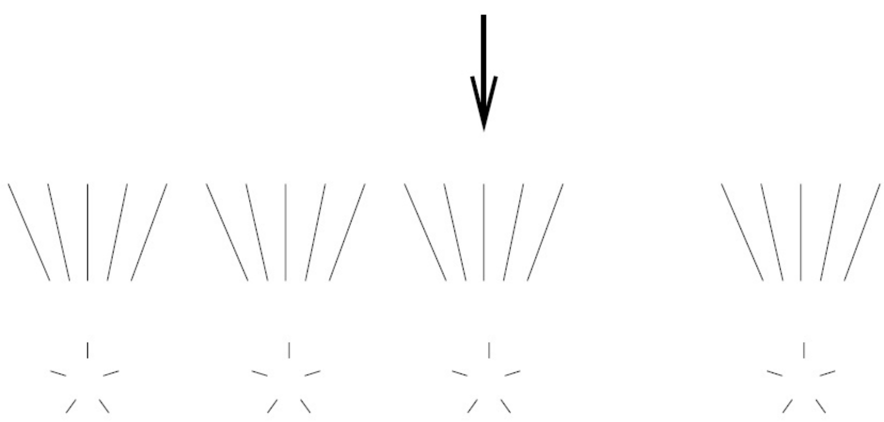
Note. In Exercise 7.3.5, it is to be shown that for any pairwise balanced design (P, B) with $X \subset P$ as a subset of points of P, the set $P \setminus X$ with the collection of blocks

$$b \setminus X \text{ for all } b \in B$$

is a pairwise balanced design (called the PBD *derived* from (P, B) by deleting the points in X). It is also to be shown that for (P, B) a projective plane of order n with ∞ as any line of B, the block design derived from (P, B) be deleting the points on ∞ is an affine plane. See the figure below. Since a projective plane of order n has $n^2 + n + 1$ points and all lines contain $n + 1$ points by Exercise 7.2.2(c) and (e), then the resulting affine plane has $n^2 + n + 1 - (n + 1) = n^2$ points and so is an affine plane of order n (see Exercise 7.1.3(e) and the definition of "order" of an affine plane).



Projective plane of order n.



Affine plane formed by deleting the line ∞.

Note. In conclusion, we can use an affine plane of order n to create a projective plane of order n (by adding a line at infinity; see Exercise 7.3.3). We can also use a projective plane of order n to create an affine plane of order n (by deleting any line, as described in Exercise 7.3.5).

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