# 7.4. Connections between Affine and Complete Sets of MOLS

**Note.** In this section, we use a complete set of mutually orthogonal latin squares of order $n$ ("MOLS($n$)") to create and affine plane of order $n$, and we use an affine plane of order $n$ to create a complete set of MOLS($n$). Recall that a complete set of MOLS($n$) is a set of $n-1$ MOLS($n$). We know that if $n$ is a power of a prime $p$, $n = p^k$, then there is a Galois field of order $n$ and so, by Theorem 6.2.2, there is a complete set of MOLS($n$). The number of MOLS($n$) that may exist for a given order $n$ is not in general known. In this section we simply assume the existence a complete set of MOLS($n$) without additional concern over the value of $n$.

**Note.** Let $L_1, L_2, \ldots, L_{n-1}$ be a complete set of MOLS($n$). Let $P = \{(i,j) \mid 1 \leq i,j \leq n\}$ and arrange these $n^2$ ordered pairs in an $n \times n$ grid ("matrix") $A$ as given in Figure 7.4.

A =

| | | | | |
|---|---|---|---|---|
| (1, n) | (2, n) | (3,n) | $\cdots$ | (n,n) |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| (1,3) | (2,3) | (3,3) | $\cdots$ | (n,3) |
| (1,2) | (2,2) | (3,2) | $\cdots$ | (n,2) |
| (1,1) | (2,1) | (3,1) | $\cdots$ | (n,1) |

Figure 7.4: Naming the cells of $A$.

**Note 7.4.A.** Let $L_1, L_2, \ldots, L_{n-1}$ be a complete set of MOLS($n$) and let set $P$ and grid $A$ be as defined above. Define a collection of subsets $B$ of $P$ where each set in $B$ had $n$ ordered pairs and is defined as:

**(1)** Each of the $n$ columns of $A$ belongs to $B$.

**(2)** Each of the $n$ rows of $A$ belongs to $B$.

**(3)** For each latin square $L_i$, each of the symbols $1, 2, \ldots, n$ determines a transversal of $A$. Place each of these $n$ transversals in $B$ (for a total of $n(n-1) = n^2 - n$ elements of $B$).

It is to be shown in Exercise 7.4.6 that $(P, B)$ is an affine plane of order $n$. The $n$ lines given in (1) form a parallel class, the $n$ lines given in (2) form a parallel class, and the $n^2 - n$ lines in (3) determined by each latin square form a parallel class. This gives a total of $(n)+(n)+(n^2-n) = n^2+n$ lines and $(1)+(1)+(n-1) = n+1$ parallel classes.

**Example 7.4.1.** Consider the complete set of MOLS(4):

$L_1 =$

| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 3 |
| 2 | 1 | 3 | 4 |
| 4 | 3 | 1 | 2 |
| 3 | 4 | 2 | 1 |

$L_2 =$

| | | | |
|---|---|---|---|
| 1 | 3 | 4 | 2 |
| 4 | 2 | 1 | 3 |
| 2 | 4 | 3 | 1 |
| 3 | 1 | 2 | 4 |

$L_3 =$

| | | | |
|---|---|---|---|
| 1 | 4 | 2 | 3 |
| 3 | 2 | 4 | 1 |
| 4 | 1 | 3 | 2 |
| 2 | 3 | 1 | 4 |

This gives matrix $A$ as

$$A = \begin{array}{|c|c|c|c|} \hline (1,4) & (2,4) & (3,4) & (4,4) \\ \hline (1,3) & (2,3) & (3,3) & (4,3) \\ \hline (1,2) & (2,2) & (3,2) & (4,2) \\ \hline (1,1) & (2,1) & (3,1) & (4,1) \\ \hline \end{array}.$$

For (1), we take each of the $n = 4$ columns of $A$ as elements of $B$, and for (2) we take each of the $n$ rows of $A$ as elements of $B$ (since these are sets, order does not matter and we list them all "horizontally"):

$$\begin{array}{llll} (1) & \{(1,4),(1,3),(1,2),(1,1)\} & (2) & \{(1,4),(2,4),(3,4),(4,4)\} \\ & \{(2,4),(2,3),(2,2),(2,1)\} & & \{(1,3),(2,3),(3,3),(4,3)\} \\ & \{(3,4),(3,3),(3,2),(3,1)\} & & \{(1,2),(2,2),(3,2),(4,2)\} \\ & \{(3,4),(4,3),(4,2),(4,1)\} & & \{(1,1),(2,1),(3,1),(4,1)\}. \end{array}$$

For (3) we use each latin square $L_1, L_2, L_3$ and each of the symbols $j \in \{1,2,3,4\}$ to determine a transversal of $A$:

$$\begin{array}{llll} L_1: & j=1,\ \{(1,4),(2,3),(3,2),(4,1)\} & L_2: & j=1,\ \{(1,4),(3,3),(4,2),(2,1)\} \\ & j=2,\ \{(2,4),(1,3),(4,2),(3,1)\} & & j=2,\ \{(4,4),(2,3),(1,2),(3,1)\} \\ & j=3,\ \{(4,4),(3,3),(2,2),(1,1)\} & & j=3,\ \{(2,4),(4,3),(3,2),(1,1)\} \\ & j=4,\ \{(3,4),(4,3),(1,2),(2,1)\}, & & j=4,\ \{(3,4),(1,3),(2,2),(4,1)\}, \end{array}$$

$$\begin{array}{ll} L_3: & j=1,\ \{(1,4),(4,3),(2,2),(3,1)\} \\ & j=2,\ \{(3,4),(2,3),(4,2),(1,1)\} \\ & j=3,\ \{(4,4),(1,3),(3,2),(2,1)\} \\ & j=4,\ \{(2,4),(3,3),(1,2),(4,1)\}. \end{array}$$

Notice that there are $n^2+n = (4)^2+(4) = 20$ lines and $n+1 = (4)+1 = 5$ parallel classes.

**Note 7.4.B.** The process of Note 7.4.A can be reversed to create a complete MOLS($n$) from an affine plane of order $n$. Let $(P, B)$ be an affine plane of order $n$. Label the $n+1$ parallel classes on $B$ (known to exist by Exercise 7.3.2) as $V, H, \pi_1, \pi_2, \ldots, \pi_{n-1}$ and label the lines in each parallel class with $1, 2, \ldots, n$??? For each parallel class $\pi_x$ we construct a latin square $L_x$ or order $n$ as follows. Let cell $(i, j)$ of $L_x$ contain the label of the line in $\pi_x$ which contains the point of intersection of line $i$ in $V$ with line $j$ in $H$ (since $V$ and $H$ are different parallel classes, then every line in $V$ intersects every line in $H$). The latin squares $L_1, L_2, \ldots, L_{n-1}$ are a complete setof MOLS($n$), as is to be verified in Exercise 7.4.7.

**Example 7.4.2.** Consider the affine plane of order 3 from Example 7.1.1(b): $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and

$$\begin{aligned} B = \{\{1,2,3\}, &\ \ \{1,4,7\}, \ \ \{1,5,9\}, \ \ \{1,6,8\}, \\ \{4,5,6\}, &\ \ \{2,5,8\}, \ \ \{2,6,7\}, \ \ \{2,4,9\}, \\ \{7,8,9\}, &\ \ \{3,6,9\}, \ \ \{3,4,8\}, \ \ \{3,5,7\}\}. \end{aligned}$$

The four parallel classes are given as the columns here. Denote these columns as $V, H, \pi_1, \pi_2$, respectively. Then label the lines (arbitrarily) with 1, 2, 3:

$$\begin{array}{cccc} 1: \ \{1,2,3\} & 1: \ \{1,4,7\} & 1: \ \{1,5,9\} & 1: \ \{1,6,8\} \\ 2: \ \{4,5,6\} & 2: \ \{2,5,8\} & 2: \ \{2,6,7\} & 2: \ \{2,4,9\} \\ 3: \ \{7,8,9\} & 3: \ \{3,6,9\} & 3: \ \{3,4,8\} & 3: \ \{3,5,7\} \\ V & H & \pi_1 & \pi_2 \end{array}$$

We now describe finding the entries of $L_1$. Starting with $(i, j) = (1, 1)$, we find the intersection of line "1" in $V$ with line "1" in $H$, which we see is symbol 1. We then find the line in $\pi_1$ that contains symbol 1 and see that it is line "1." So cell $(1,1)$ of $L_1$ contains symbol 1. Similarly, we have:

| $(i,j)$ | line $i$ of $V$ | line $j$ of $H$ | intersection | line of $\pi_1$ containing intersection | symbol of line in $\pi_1$ |
|---|---|---|---|---|---|
| $(1,1)$ | $\{1,2,3\}$ | $\{1,4,7\}$ | 1 | $\{1,5,9\}$ | 1 |
| $(1,2)$ | $\{1,2,3\}$ | $\{2,5,8\}$ | 2 | $\{2,6,7\}$ | 2 |
| $(1,3)$ | $\{1,2,3\}$ | $\{3,6,9\}$ | 3 | $\{3,4,8\}$ | 3 |
| $(2,1)$ | $\{4,5,6\}$ | $\{1,4,7\}$ | 4 | $\{3,4,8\}$ | 3 |
| $(2,2)$ | $\{4,5,6\}$ | $\{2,5,8\}$ | 5 | $\{1,5,9\}$ | 1 |
| $(2,3)$ | $\{4,5,6\}$ | $\{3,6,9\}$ | 6 | $\{2,6,7\}$ | 2 |
| $(3,1)$ | $\{7,8,9\}$ | $\{1,4,7\}$ | 7 | $\{2,6,7\}$ | 2 |
| $(3,2)$ | $\{7,8,9\}$ | $\{2,5,8\}$ | 8 | $\{3,4,8\}$ | 3 |
| $(3,3)$ | $\{7,8,9\}$ | $\{3,6,9\}$ | 9 | $\{1,5,9\}$ | 1 |

This gives $L_1$ as:

$$L_1 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 1 & 2 \\ \hline 2 & 3 & 1 \\ \hline \end{array}\,.$$

The entries have been entered in $L_1$ starting in the upper left hand corner (and working to the right, as usual). In the next section we will find it useful to create latin squares by starting in the lower left hand corner and working upward. Lindner and Rodger present $L_1$ and $L_2$ as we do here in these notes, but they also give alternative versions in which their $j$th column (read from bottom to top) is our $j$th row (read from left to right).

For $L_2$:

| $(i,j)$ | line $i$ of $V$ | line $j$ of $H$ | intersection | line of $\pi_2$ containing intersection | symbol of line in $\pi_2$ |
|---|---|---|---|---|---|
| $(1,1)$ | $\{1,2,3\}$ | $\{1,4,7\}$ | 1 | $\{1,6,8\}$ | 1 |
| $(1,2)$ | $\{1,2,3\}$ | $\{2,5,8\}$ | 2 | $\{2,4,9\}$ | 2 |
| $(1,3)$ | $\{1,2,3\}$ | $\{3,6,9\}$ | 3 | $\{3,5,7\}$ | 3 |
| $(2,1)$ | $\{4,5,6\}$ | $\{1,4,7\}$ | 4 | $\{2,4,9\}$ | 2 |
| $(2,2)$ | $\{4,5,6\}$ | $\{2,5,8\}$ | 5 | $\{3,5,7\}$ | 3 |
| $(2,3)$ | $\{4,5,6\}$ | $\{3,6,9\}$ | 6 | $\{1,6,8\}$ | 1 |
| $(3,1)$ | $\{7,8,9\}$ | $\{1,4,7\}$ | 7 | $\{3,5,7\}$ | 3 |
| $(3,2)$ | $\{7,8,9\}$ | $\{2,5,8\}$ | 8 | $\{1,6,8\}$ | 1 |
| $(3,3)$ | $\{7,8,9\}$ | $\{3,6,9\}$ | 9 | $\{2,4,9\}$ | 2 |

This gives $L_1$ and $L_2$ as:

$$L_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 1 & 2 \\ \hline 2 & 3 & 1 \\ \hline \end{array} \qquad L_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array}.$$

**Note.** Notes 7.4.A and 7.4.B show that an affine plane of order $n$ corresponds to a complete set of MOLD($n$), and conversely. This gives the following.

**Theorem 7.4.3.** An affine plane of order $n$ (and therefore a projective plane of order $n$) is equivalent to a complete set of MOLS($n$).

**Note.** We know by Theorem 6.2.2 that a complete set of MOLS($n$) exists for all $n$ a power of a prime $p$ greater than 2, $n = p^\alpha > 2$. Therefore there exists an affine plane (by Note 7.4.A) and so a projective plane (by Exercise 7.3.3) for these orders. These are the only orders for which affine planes are known to exist (according to Lindner and Rodger, page 164). Some progress has been made in eliminating the existence of affine planes of certain orders. In R. H. Bruck and H. J. Ryser, "The Non-Existence of Certain Finite Projective Planes," *Canadian Journal of Mathematics*, **1**, 88–93 (1949), available online on the Cambridge University Press website, the following is shown.

**Theorem 7.4.4.** Let $n \equiv 1$ or $2 \pmod 4$ and let the square-free part of $n$ contain at least one prime factor $p \equiv 3 \pmod 4$. Then there does not exist an affine plane of order $n$.

**Note.** Theorem 7.4.4 rules out an affine plane of order 6 (since 6 is square free and 3 divides 6). It rules out affine planes of order 14 (since 14 is square free and 7 divides 14, where $7 \equiv 3 \pmod 4$) and order 22 (since 22 is square free and 11 divides 22, where $11 \equiv 3 \pmod 4$). The case of an order 10 projective plane (and, equivalently, an order 10 affine plane) was addressed in the 1980s with an extensive computer search. The story of this search is described in Clement Lam's "The Search for a Finite Projective Plane of Order 10," *The American Mathematical Monthly*, **98**(4), 305–318 (1991). A copy is available online on the MAA webpage. This leaves the smallest unsettled case for the existence of an affine plane of order

$n$ as $n = 12$ (we have that 3, 4, 5, 7, 8, 9, and 11 are primes or primes powers, so that by Theorem 6.2.2 a complete set of MOLS($n$) exists and hence an affine plane of these orders exist by Note 7.4.A; 6 is covered by Theorem 7.4.4, and 10 is covered in the computer search). So much work is left to be done in addressing the existence/nonexistence of finite affine planes! Beyond Theorem 7.4.4, the computer search for $n = 10$, and the constructions for prime powers of Theorem 6.2.2 and Note 7.4.A, there seems to be no other definitive results. This leads to the following.

**Open Problem.** Does there exist an affine plane of non-prime power order?

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