# 7.5. Coordinatizing the Affine Plane

**Note.** In this section, we associate $x$-coordinates and $y$-coordinates with the points in an affine plane. This allows us to define the slope of a line and to represent lines in the form $y = mx + b$.

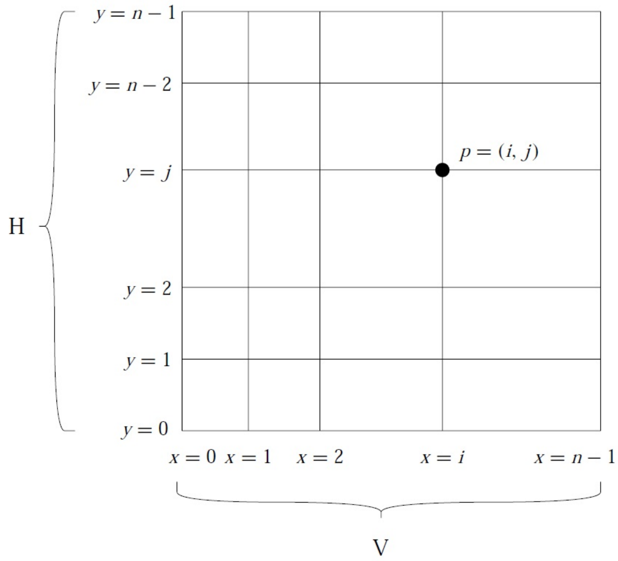
**Note/Definition.** Let $(P, B)$ be an affine plane of order $n$. Then, by Exercise 7.1.3(c), (e), (f) (respectively), every line contains $n$ points, $|P| = n^2$, and $|B| = n^2 + n$. By Exercise 7.3.2 there are $n+1$ parallel classes. Label the parallel classes $V, H, \pi_1, \pi_2, \ldots, \pi_{n-1}$ and label the lines in $V$ and $H$ as:

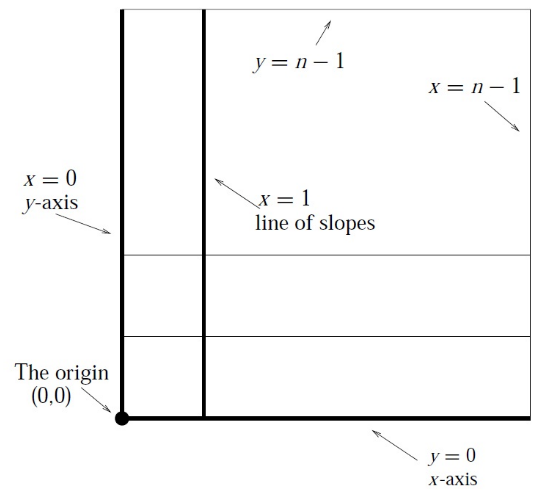
$$V : x = 0, x = 1, x = 2, \ldots, x = n - 1$$

$$H : y = 0, y = 1, y = 2, \ldots, y = n - 1.$$

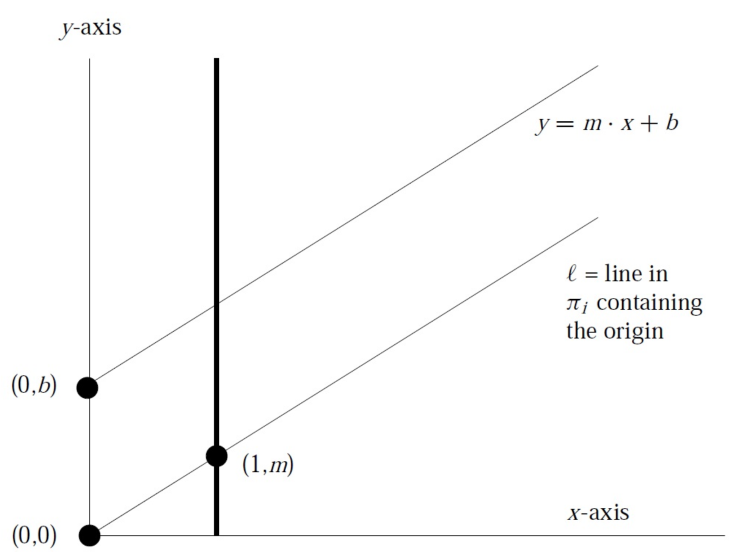
Point $p \in P$ has *coordinates* $(i, j)$ if and only if $p$ belongs to line $x = i$ and the line $y = j$. Notice that by Exercise 7.1.3(d) every point is on $n + 1$ lines, so any given point must lie on exactly one line from each parallel class (of course a point can't lie on two lines in the same parallel class).

**Note/Definition.** Notice that by the definition of "coordinates," the line $x = i$ contains precisely the points with first coordinate $i$, and the line $y = j$ contains precisely the points with second coordinate $j$. The line $y = 0$ is the *$x$-axis*, the line $x = 0$ is the *$y$-axis*, the line $x = 1$ is the *line of slopes*, and the point $(0, 0)$ is the *origin*. See the figure below.




**Note/Definition.** As described above, any given point must lie on exactly one line from each parallel class. So each parallel class $\pi_i$ has exactly one line $\ell$ containing the origin. Since $\ell \notin V$ and $\ell \notin H$, then $\ell$ must intersect each line in the parallel class $x = 1$. Let $(1, m)$ be the point of intersection of $\ell$ and the line in parallel class $x = 1$. Then each line in parallel class $\pi_i$ has *slope* $m$. A line in parallel class $V$ has *no slope* and a line in parallel class $H$ has slope 0. See the figure below.



**Definition.** A line in parallel class $\pi_i$ can be labeled as $y = mx + b$ where $m$ is the slope and $(0, b)$ is the point of intersection of the line with the $y$-axis. The equation $y = mx + b$ is the *point slope formula* for the line.

**Example 7.5.1.** Consider again the parallel classes of the affine plane of order 2 from Examples 7.1.1(b) and 7.4.2:

$$\begin{array}{cccc} \{1,2,3\} & \{1,4,7\} & \{1,5,9\} & \{1,6,8\} \\ \{4,5,6\} & \{2,5,8\} & \{2,6,7\} & \{2,4,9\} \\ \{7,8,9\} & \{3,6,9\} & \{3,4,8\} & \{3,5,7\} \\ V & H & \pi_1 & \pi_2 \end{array}$$

We label the lines of $H$ and $V$ as needed:

$$\begin{array}{cc} x = 0: \ \{1,2,3\} & y = 0: \ \{1,4,7\} \\ x = 1: \ \{4,5,6\} & y = 1: \ \{2,5,8\} \\ x = 2: \ \{7,8,9\} & y = 2: \ \{3,6,9\} \\ V & H \end{array}$$

We find the coordinates of the 9 points (represented by symbols $1, 2, \ldots, 9$) by finding the two lines in $V$ and $H$ that contain the point: $1 = (0, 0)$, $2 = (1, 0)$, $3 = (2, 0)$, $4 = (0, 1)$, $5 = (1, 1)$, $6 = (2, 1)$, $7 = (0, 2)$, $8 = (1, 2)$, and $9 = (2, 2)$. Now we find the point-slope formula for the remaining six lines. For parallel class $\pi_1$, we find the line containing the symbols $(0, 0) = 1$. This is line $\ell = \{1, 5, 9\} \in \pi_1$. The intersection of line $\ell = \{1, 5, 9\}$ with line $x = 1 : \{2, 5, 8\}$ is the symbol 5 which represents the point $(1, 1) = (1, m)$ and so $m = 1$ is the slope of all lines in $\pi_1$. So lines in parallel class $\pi_1$ are of the form $y = 1x + b$ for some $b$. For parallel class $\pi_2$, we find the line containing the symbols $(0, 0) = 1$. This is line $\ell = \{1, 6, 8\} \in \pi_2$.

The intersection of line $\ell = \{1, 6, 8\}$ with line $x = 1 : \{2, 5, 8\}$ is the symbol 8 which represents the point $(1, 2) = (1, m)$ and so $m = 2$ is the slope of all lines in $\pi_2$. So lines in parallel class $\pi_2$ are of the form $y = 2x + b$ for some $b$. Next, the value of $b$ for a given line results from the symbol of intersection of the line with the $y$-axis $x = 0 : \{1, 4, 7\}$ expressed in coordinates of the form $(0, b)$. For the six lines we have:

| $\pi_i$ | line | $y$-axis | symbol of intersection | $b$ | $y = mx + b$ |
|---|---|---|---|---|---|
| $\pi_1$ | $\{1, 5, 9\}$ | $\{1, 4, 7\}$ | $1 = (0, 0)$ | 0 | $y = 1x + 0$ |
| $\pi_1$ | $\{2, 6, 7\}$ | $\{1, 4, 7\}$ | $7 = (0, 2)$ | 2 | $y = 1x + 2$ |
| $\pi_1$ | $\{3, 4, 8\}$ | $\{1, 4, 7\}$ | $4 = (0, 1)$ | 1 | $y = 1x + 1$ |
| $\pi_2$ | $\{1, 6, 8\}$ | $\{1, 4, 7\}$ | $1 = (0, 0)$ | 0 | $y = 2x + 0$ |
| $\pi_2$ | $\{2, 4, 9\}$ | $\{1, 4, 7\}$ | $4 = (0, 1)$ | 1 | $y = 2x + 1$ |
| $\pi_2$ | $\{3, 5, 7\}$ | $\{1, 4, 7\}$ | $7 = (0, 2)$ | 2 | $y = 2x + 2$ |

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