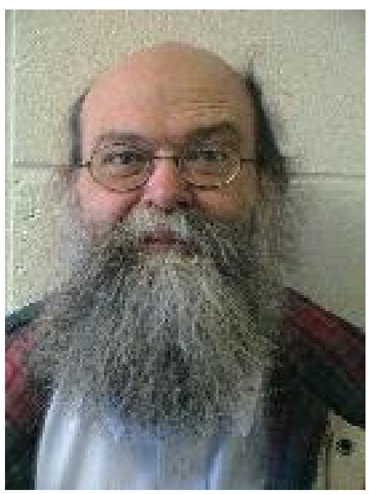
Chapter 8. Intersections of Steiner Triple Systems

8.1. Teirlinck's Algorithm

Note. In this section, we show that for every Steiner triple system on a set S, there is a second Steiner triple system on set S which has no triples in common with the first system. The results of the section appear in Luc Teirlinck's "On Making Two Steiner Triple Systems Disjoint," *Journal of Combinatorial Theory, Series A*, **23**, 349–350. This paper can be views online on the ScienceDirect website. Luc Teirlinck is another of the Auburn University combinatorics faculty. He was a colleague of Curt Lindner and Chris Rodger and served on my master's thesis committee in the 1980s. He is currently retired is an emeritus professor.



From Professor Tierlinck's faculty webpage (accessed 8/17/2022)

Definition. Two Steiner triple systems on the same set S, (S, T_1) and (S, T_2), are *disjoint* if they have no triples in common; that is, if $T_1 \cap T_2 = \varnothing$.

Example 8.1.1. An example of two disjoint $STS(7)$ on set $S = \{1,2,3,4,5,6,7\}$ have triples:

$$T_1 = \left\{ \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \\ 5 & 6 & 1 \\ 6 & 7 & 2 \\ 7 & 1 & 3 \end{array} \right. \qquad T_2 = \left\{ \begin{array}{ccc} 1 & 2 & 5 \\ 2 & 4 & 4 \\ 4 & 5 & 3 \\ 5 & 6 & 7 \\ 6 & 3 & 1 \\ 3 & 7 & 2 \\ 7 & 1 & 4 \end{array} \right.$$

Note. Informally, an isomorphism of two mathematical objects is a bijection (a one-to-one and onto) mapping from object to the other that preserves structure. For example, the structure of a a graph is adjacency so, more formally, a graph isomorphism is a bijection between the two vertex sets that preserves adjacency (see my online notes for Introduction to Graph Theory (MATH 4347/5347) on Section 1.2. Subgraphs, Isomorphic Graphs). The structure of a vector space is the interaction of vectors and scalars (that is, linear combinations), so a vector space isomorphism is a bijection between the sets of vectors which preserves linear combinations (see my online notes for Linear Algebra (MATH 2010) on Section 3.3. Coordinatization of Vectors; the vector spaces must have a common scalar field for an isomorphism to be defined). In a group, the structure is the binary operation, and so forth. The structure of a Steiner triple system is the collection of triples. Hence, we have the following definition.

Definition. Let (S, T_1) and (S, T_2) be Steiner triple systems. If there is a permutation α on S (that is, α is a bijection mapping $S \to S$) such that

$$T_1\alpha = \{\{x\alpha, y\alpha, z\alpha\} \mid \{x, y, z\} \in T_1\} = T_2,$$

then (S, T_1) and (S, T_2) are *isomorphic*. Mapping α is a Steiner triple system *isomorphism*.

Example 8.1.2. The two Steiner triple systems of Example 8.1.2 are isomorphic with isomorphism $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 5 & 6 & 3 & 7 \end{pmatrix} = (3, 4, 5, 6)$. That is, we have $T_1\alpha = T_2$.

Note. Recall that every permutation in the symmetric group S_n can be written as a product of transpositions (that is, cycles of length two). See my online notes for Introduction to Modern Algebra (MATH 4127/5127) on Section II.9. Orbits, Cycles, Alternating Groups; notice Corollary 9.12. We seek an isomorphism α that allows us to find a disjoint "mate" to a given Steiner triple system, as in the previous example. We will do so by constructing α as a product of transpositions.

Note 8.1.A. Denote the transposition that interchanges a and b by (a, b) (notice that Lindner and Rodger omit the comma in their notation). Let (S, T_1) and (S, T_2) be any two Steiner triple systems of order n and let $\{c, d, e\} \in T_1 \cap T_2$. Define the sets

$$\begin{aligned} A(c) &= \{a \mid \{a,x,y\} \in T_1 \text{ and } \{c,x,y\} \in T_2 \setminus \{c,d,e\}\} \\ B(c) &= \{b \mid \{b,z,w\} \in T_2 \text{ and } \{c,z,w\} \in T_1 \setminus \{c,d,e\}\}. \end{aligned}$$

Hence set $A(c)$ is related to the triples of T_2 which, when c is replaced by a, are shared by both triple systems (with the exception of triple $\{c,d,e\}$, if applicable). Set $B(c)$ is related to the triples of T_1 which, when c is replaced by b, are shared by both triple systems (with the exception of triple $\{c,d,e\}$, if applicable). The plan is to make some interchanges of elements of S in order to produce disjoint Steiner triple systems. Define the *spread* of c with respect to $\{c,d,e\}$ as $S(c) = \{c,d,e\} \cup A(c) \cup B(c)$. This is illustrated as follows, where $t = (n-3)/2$ so that either triple system has $t+1 = (t-1)/2$ triples containing c; there are t such triples in addition to the shared triple $\{c,d,e\}$. Notice that if $\{c,d,e\}$, $A(c)$, and $B(c)$ are pairwise disjoint then $|S(c)| = 3 + 2(n-3)/2 = n$ and $S(c) = S$, and conversely.

$$\begin{array}{ccc} \underline{\quad T_1 \quad} & & \underline{\quad T_2 \quad} \\ \begin{array}{ccc} c & d & e \\ c & z_1 & w_1 \\ c & z_2 & w_2 \\ & \vdots & \\ c & z_t & w_t \end{array} & t = (n-3)/2 & \begin{array}{ccc} c & d & e \\ c & x_1 & y_1 \\ c & x_2 & y_2 \\ & \vdots & \\ c & x_t & y_t \end{array} \end{array}$$

$$A = \begin{array}{|c|cc} a_1 & x_1 & y_1 \\ a_2 & x_2 & y_2 \\ & \vdots & \\ a_t & x_t & y_t \end{array} \qquad B = \begin{array}{|c|cc} b_1 & z_1 & w_1 \\ b_2 & z_2 & w_2 \\ & \vdots & \\ b_t & z_t & w_t \end{array}$$

Example 8.1.3. To illustrate spread, consider the two Steiner triple systems of order 15 of (S, T_1) and $(S, T_2)$with the following triples (given here in an obvious notation):

T_1 :

1	2	3	2	4	7	3	5	15	10	12	14	9	4	12
1	4	11	2	5	8	3	6	10	11	13	15	10	11	5
1	5	12	2	6	9	3	7	11	4	5	13	11	12	6
1	6	13	2	10	13	3	8	12	5	6	14	12	13	7
1	7	14	2	11	14	3	9	13	6	7	15	13	14	8
1	8	15	2	12	15	4	6	8	7	8	10	14	15	9
1	9	10	3	4	14	**5**	**7**	**9**	8	9	11	15	10	4

T_2 :

1	2	3	2	4	7	3	5	13	10	12	14	15	10	6
1	4	15	2	5	8	3	6	14	11	13	15	4	5	11
1	5	10	2	6	9	3	7	15	10	11	7	5	6	12
1	6	11	2	10	13	3	8	10	11	12	8	6	7	13
1	7	12	2	11	14	3	9	11	12	13	9	7	8	14
1	8	13	2	12	15	4	6	8	13	14	4	8	9	15
1	9	14	3	4	12	**5**	**7**	**9**	14	15	5	9	4	10

(a) Notice that triple $\{5, 7, 9\}$ is common to both STSs, $\{5, 7, 9\} \in T_1 \cap T_2$. We take $c = 5$. With respect to this triple we have (reading left to right and repeating values; the relevant triples are in blue fonts above) $A(c) = A(5) = \{9, 5, , 9, 1, 11\}$. Similarly, $B(c) = B(5) = \{7, 5, 7, 14, 3, 7\}$. So the spread of $c = 5$ is $S(5) = \{5, 7, 9\} \cup A(5) \cup B(5) = \{1, 3, 5, 7, 9, 11, 14\}$.

(b) We have $\{1, 2, 3\} \in T_1 \cap T_2$ and take $c = 3$. With respect to the common triple we have $S(3) = \{1, 2, 3\} \cup A(5) \cup B(5) = \{5, 7, 9\} \cup \{4, 5, 6, 7, 8, 9\} \cup$

$\{10, 11, 12, 13, 14, 15\} = \{1, 2, \ldots, 15\} = S$.

Note. Our ultimate goal in this section is to produce a disjoint mate for a given Steiner triple system. In this direction, we consider "The Reduction Algorithm" that allows us to consider two STSs (S, T_1) and (S, T_2) which share a triple, and modify the triples in T_2 is such a way that the result is a new STS with fewer triples in common with T_1 than T_2 had. This algorithm appears in Jean Doyen, "Constructions of Disjoint Steiner Triple Systems," *Proceedings of the American Mathematical Society*, **32**, 409–416 (1972) (available on the AMS website; accessed 8/19/2022). We state the result of the algorithm as a theorem and use the algorithm itself as the proof for the theorem.

Theorem 8.1.A. The Reduction Algorithm. Let (S, T_1) and (S, T_2) be any two STS(n)s and suppose that $\{1, 2, 3\} \in T_1 \cap T_2$ and $|S(3)| < n$. Then there exists a transposition α such that $T_1 \cap T_2\alpha \subseteq T_1 \cap T_2$ and $|T_1 \cap T_2\alpha| < |T_1 \cap T_2|$.

Example 8.1.8(a). Let (S, T_1) and (S, T_2) be the two STS(9)s with the triples:

$$T_1 = \left| \begin{array}{ccc|ccc|ccc|ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 & 4 & 7 & 1 & 5 & 9 & 1 & 6 & 8 \\ 4 & 5 & 6 & 2 & 5 & 8 & 2 & 6 & 7 & 2 & 4 & 9 \\ 7 & 8 & 9 & 3 & 6 & 9 & 3 & 4 & 8 & 3 & 5 & 7 \end{array} \right|$$

$$T_2 = \left| \begin{array}{ccc|ccc|ccc|ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 & 4 & 7 & 1 & 9 & 5 & 1 & 8 & 6 \\ 4 & 9 & 8 & 2 & 9 & 6 & 2 & 8 & 7 & 2 & 4 & 5 \\ 7 & 6 & 5 & 3 & 8 & 5 & 3 & 4 & 6 & 3 & 9 & 7 \end{array} \right|$$

Then $T_1 \cap T_2 = \{\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 9\}, \{1, 6, 8\}\}$. We consider $\{1, 2, 3\} \in$

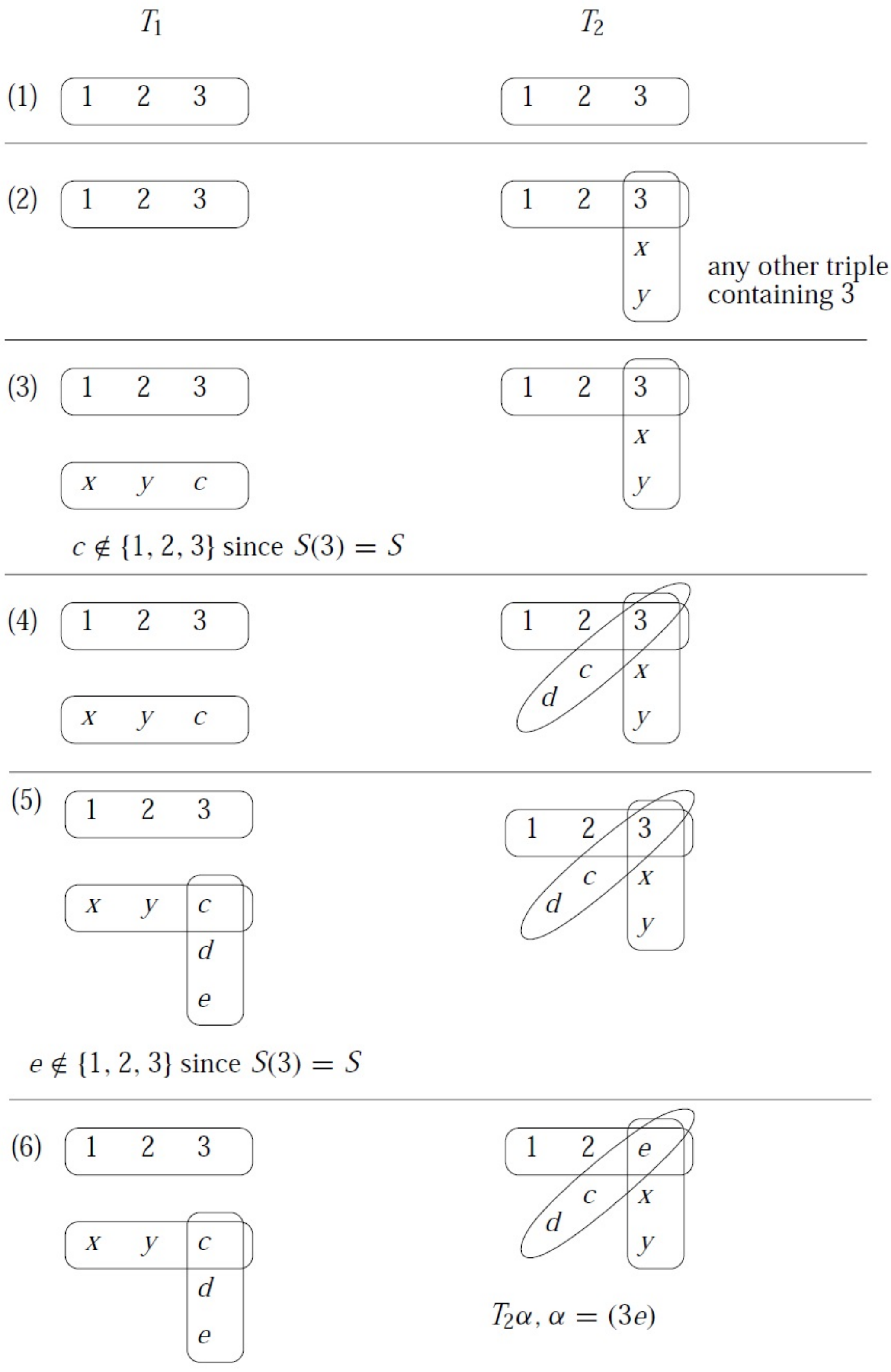
$T_1 \cap T_2$. With $c = 3$ we have (as illustrated in blue) $A(c) = A(3) = \{2, 5, 8\}$ and (illustrated in red) $B(c) = B(3) = \{2, 9, 6\}$. Therefore $S(c) = S(3) = \{1, 2, 3\} \cup A(3) \cup B(3) = \{1, 2, 3, 5, 6, 8, 9\}$. Since $4 \not\in S(3)$, if we take $\alpha = (3, 4)$ then (as can be checked; see Exercise 8.1.9) $T_1 \cap T_2 = (T_1 \cap T_2) \setminus I = \{\{1, 5, 9\}, \{1, 6, 8\}\}$. So T_1 and $T_2\alpha$ share fewer triples than the original T_1 and T_2. Notice that in Exercise 8.1.8(b), c is taken to be 1 (where we still start with triple $\{1, 2, 3\}$ as the shared triple) and $\alpha = (1, 4)$ from which T_1 and $T_2\alpha$ then only share one triple.

Note 8.1.B. *If* we can repeatedly apply the Reduction Algorithm (Theorem 8.1.A) to produce a sequence of transpositions (and hence a permutation) on S such that $T_1 \cap T_2\alpha_1\alpha_2 \cdots \alpha_j = \varnothing$, then the result is a disjoint mate of Steiner triple system (S, T_1). But a case can arise in which the hypotheses of the Reduction Algorithm are not satisfied. This can happen when every element of every triple of $T_1 \cap T_2\alpha_1\alpha_2 \cdots \alpha_j$ has spread equal to all of S (so that the element $c \in S$ cannot be chosen such that $|S(c)| < n$). It is Teirlinck's Algorithm that addresses this case.

Theorem 8.1.B. Teirlinck's Algorithm. Let (S, T_1) and (S, T_2) be any two STS(n)s and suppose that $\{1, 2, 3\} \in T_1 \cap T_2$ and $S(3) = S$. Then there exists a transposition α such that $T_1 \cap T_2\alpha$ contains a triple t and an element $e \in t$ such that $|S(e)| < n$ (where this spread is with respect to triple t) and $|T_1 \cap T_1\alpha| \leq |T_1 \cap T_2|$.

Note. Teirlinck's Algorithm is illustrated graphically on the next page (from page 173 of the textbook).

Algorithm Let $\{1, 2, 3\} \in T_1 \cap T_2$.



Then $\{c, d, e\} \in T_1 \cap T_2\alpha$ and $|S(e)| < n$. The fact that $|T_1 \cap T_2\alpha| \leq |T_1 \cap T_2|$ is left as an exercise.

Note. Teirlink's Algorithm is illustrated graphically for a specific example in Example 8.1.11 and similarly in Exercises 8.1.12 and 8.1.13.

Note. Given two STSs, (S, T_1) and (S, T_2), we can iteratively apply the Reduction Algorithm (Theorem 8.1.A) to create a sequence of new STSs $(S, T_2\alpha_1)$, $(S, T_2\alpha_1\alpha_2)$, $\ldots (S, T_2\alpha_1\alpha_2\cdots\alpha_k)$ where each α_i is a transposition and

$$|T_1 \cap T_2| > |T_1 \cap T_2\alpha_1| > |T_1 \cap T_2\alpha_1\alpha_2| > \cdots > |T_1 \cap T_2\alpha_1\alpha_2\cdots\alpha_j|.$$

Either this sequence terminates when $T_1 \cap T_2\alpha_1\alpha_2\cdots\alpha_j = \varnothing$ (in which case we have an isomorphic disjoint mate to (S, T_1)) or we "get stuck" when every element of every triple of $T_1 \cap T_2\alpha_1\alpha_2\cdots\alpha_j$ has spread equal to all of S (see Note 8.1.B). If we do get stuck, then we can apply Teirlinck's Algorithm to introduce another transposition α_{j+1} (say) so that $|T_1 \cap T_2\alpha_1\alpha_2\cdots\alpha_j| \geq ||T_1 \cap T_2\alpha_1\alpha_2\cdots\alpha_j\alpha_{j+1}|$ and we are "unstuck" (that is, $|S(e)| < n$ with respect to some triple t). Continuing in this way of applying the Reduction Algorithm and Tierlinck's Algorithm (to the appropriate cases), we get the following.

Theorem 8.1.15. Let (S, T_1) and (S, T_2) be any two Steiner triple systems of order n. Then there exist transpositions $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $T_1 \cap T_2\alpha_1\alpha_2\cdots\alpha_k = \varnothing$.

Note. We now see that any given any Steiner triple system (S, T_1), there is a disjoint mate of the Steiner triple system that is isomorphic to (S, T_1). We simply take $T_2 = T_1$ and apply Theorem 8.1.15. The isomorphism is given by the product of the transpositions. Formally, we have the following.

Corollary 8.1.16. Every Steiner triple system has an isomorphic disjoint mate.

Note. As a final comment, notice the very constructive nature of the ultimate proof of Corollary 8.1.16. This is a common approach in discrete math (less so in some other areas of math).

Revised: 8/19/2022