Appendix A. Cyclic Steiner Triple Systems

Note. In this Appendix we give solutions to Heffter's First and Second Difference Problems, which we encountered in Section 1.7. Cyclic Steiner Triple Systems. We saw in that section that solutions to Heffter's Difference Problems can be used to construct cyclic Steiner triple systems (see Theorem 1.7.6). These problems were posed by Lothar Heffter (June 11, 1862–January 1, 1962) in "Ueber Tripelsysteme," *Mathematische Annalen* **49**(1), 101–112 (1897). This paper is available online on the ARchive.org webpage. He did not solve the problem, however. It was solved by Rose Peltesohn (May 16, 1913–March 21, 1998) in "Eine Lösung der beiden Heffterschen Differenzenprobleme [A Solution to the Two Heffter Difference Problems]," *Compositio Mathematica*, **6** 251–257 (1939). This paper is available online on the Numdam website, but the result is also given in Appendix A of the text book.

Lothar Heffter

Rose Peltesohn

These images are from the German Wikipedia website on Lothar Heffter and the Wikipedia webpage on Rose Peltesohn. These websites were all accessed 5/12/2022.

Note. Recall that Heffter's First and Second Difference Problems state:

(1) Let $v = 6n + 1$. Is it possible to partition the set $\{1, 2, \ldots, 3n\}$ into difference triples?

(2) Let $v = 6n + 3$. Is it possible to partition the set $\{1, 2, \ldots, 3n + 1\} \setminus \{2n + 1\}$ into difference triples?

Heffter's First Difference Problem deals with partitioning the set $\{1, 2, \ldots, (v - 1)/2\}$, and Heffter's Second Difference Problem deals with partitioning the set $\{1, 2, \ldots, (v - 1)/2\} \setminus \{v/3\}$.

Note A.1. First, we consider seven small cases. Unlike the Appendix in the text book, we include the $v/3$ difference (separately) to make checking the solutions easier. Notice that we should have all numbers $1, 2, \ldots, (v - 1)/2$ present in each case, and that $x + y = \pm z \pmod{v}$ in each case. This is straightforward (but a little tedious) to confirm here.

$\underline{v = 7:}$ $\{1, 2, 3\}$.

$\underline{v = 13:}$ $\{1, 3, 4\}$, and $\{2, 5, 6\}$.

$\underline{v = 15:}$ $\{1, 3, 4\}$, $\{2, 6, 7\}$, and $\{5\}$.

$\underline{v = 19:}$ $\{1, 5, 6\}$, $\{2, 8, 9\}$, and $\{3, 4, 7\}$.

$\underline{v = 27:}$ $\{1, 12, 13\}$, $\{2, 5, 7\}$, $\{3, 8, 11\}$, $\{4, 6, 10\}$, and $\{9\}$.

$\underline{v = 45:}$ $\{1, 11, 12\}$, $\{2, 17, 19\}$, $\{3, 20, 22\}$, $\{4, 10, 14\}$, $\{5, 8, 13\}$, $\{6, 18, 21\}$, and $\{7, 9, 16\}$, and $\{15\}$.

<u>$v = 63$:</u> $\{1, 15, 16\}$, $\{2, 27, 29\}$, $\{3, 25, 28\}$, $\{4, 14, 18\}$, $\{5, 26, 31\}$, $\{6, 17, 23\}$, $\{7, 13, 20\}$, $\{8, 11, 19\}$, $\{9, 24, 30\}$, $\{10, 12, 22\}$, $\{21\}$.

Note A.2. Suppose $v \equiv 1 \pmod{18}$ and $v \geq 37$. Say $v = 18s + 1$ where $s \geq 2$. Notice that $(3r+1)+(4s-r+1) = (4s+2r+2)$, $(3r+2)+(8s-r) = (8s+2r+2)$, $(3r+3)+(6s-2r-1) = (6s+r+2)$, and $(3s)+(3s+1) = (6s+1)$ so each of the triples given in the first column of the following table is, in fact, a difference triple. We need to check that all numbers $1, 2, \ldots, (v-1)/2$ (or their negatives modulo v; see Note 1.7.A) are present. Here $(v-1)/2 = 9s$.

$\{3r+1, 4s-r+1, 4s+2r+2\}$ for $0 \leq r \leq s-1$	$1, 4, \ldots, 3s-2; 1 \pmod 3$ $3s+2, 3s+3, \ldots, 4s+1$ $4s+2, 4s+4, \ldots, 6s$ even
$\{3r+2, 8s-r, 8s+2r+2\}$ for $0 \leq r \leq s-1$	$2, 5, \ldots, 3s-1; 2 \pmod 3$ $7s+1, 7s+2, \ldots, 8s$ $8s+2, 8s+4, \ldots, 10s$ even$*$
$\{3r+3, 6s-2r-1, 6s+r+2\}$ for $0 \leq r \leq s-2$	$3, 6, \ldots, 3s-3; 0 \pmod 3$ $4s+3, 4s+5, \ldots, 6s-1$ odd $6s+2, 6s+3, \ldots, 7s$
$\{3s, 3s+1, 6s+1\}$	$3s, 3s+1, 6s+1$

*Notice that all differences of Heffter's Difference Problems are between 1 and $(v-1)/2 = 9s$, so any difference larger than $9s$ can be replaced with its negative $\pmod v$, as stated in Note 1.7.A. Hence, if s is even, then the differences $8s+2, 8s+4, \ldots, 10s$ correspond to the differences of $8s+2, 8s+4, \ldots, 9s, v-(9s+2), v-(9s+4), \ldots, v-(10s)$, which equal $8s+2, 8s+4, \ldots, 9s, 9s-1, 9s-3, \ldots 8s+1$ (respectively) or, in order, $8s+1, 8s+2, 8s+3, \ldots, 9s$. If s is odd, then the differences

$8s+2, 8s+4, \ldots, 10s$ correspond to the differences of $8s+2, 8s+4, \ldots, 9s-1, v-(9s+1), v-(9s+3), \ldots, v-(10s)$, which equal $8s+2, 8s+4, \ldots, 9s-1, 9s, 9s-2, 9s-4, \ldots 8s+1$ (respectively) or, in order, $8s+1, 8s+2, 8s+3, \ldots, 9s$. Whether s is even or odd, the differences $8s+1, 8s+2, \ldots, 9s$.

Note A.3. Suppose $v \equiv 7 \pmod{18}$ and $v \geq 25$. Say $v = 18s+7$ where $s \geq 1$. Notice that $(3r+1)+(8s-r+3) = (8s+2r+4)$, $(3r+2)+(6s-2r+1) = (6s+r+3)$, $(3r+3)+(4s-r+1) = (4s+2r+4)$, and $(3s+1)+(4s+2) = (7s+3)$ so each of the triples given in the first column of the following table is, in fact, a difference triple. We need to check that all numbers $1, 2, \ldots, (v-1)/2$ (or their negatives modulo v; see Note 1.7.A) are present. Here $(v-1)/2 = 9s+3$.

$\{3r+1, 8s-r+3, 8s+2r+4\}$ for $0 \leq r \leq s-1$	$1, 4, \ldots, 3s-2; 1 \pmod{3}$ $7s+4, 7s+5, \ldots, 8s+3$ $8s+4, 8s+6, \ldots, 10s+2$ even$*$
$\{3r+2, 6s-2r+1, 6s+r+3\}$ for $0 \leq r \leq s-1$	$2, 5, \ldots, 3s-1; 2 \pmod{3}$ $4s+3, 4s+5, \ldots, 6s+1$ odd $6s+3, 6s+4, \ldots, 7s+2$
$\{3r+3, 4s-r+1, 4s+2r+4\}$ for $0 \leq r \leq s-1$	$3, 6, \ldots, 3s; 0 \pmod{3}$ $3s+2, 3s+3, \ldots, 4s+1$ $4s+4, 4s+6, \ldots, 6s+2$ even
$\{3s+1, 4s+2, 7s+3\}$	$3s+1, 4s+2, 7s+3$

*Notice that all differences of Heffter's Difference Problems are between 1 and $(v-1)/2 = 9s+3$, so any difference larger than $9s+3$ can be replaced with its negative (mod v), as stated in Note 1.7.A. Hence, if s is even, then the differences $8s+4, 8s+6, \ldots, 10s+2$ correspond to the differences of $8s+4, 8s+6, \ldots, 9s+2, v-$

$(9s+4), v-(9s+6), v-(9s+8), \ldots, v-(10s+2)$, which equal $8s+4, 8s+6, \ldots, 9s+2, 9s+3, 9s+1, 9s-1, \ldots, 8s+5$ (respectively) or, in order, $8s+4, 8s+5, 8s+6 \ldots, 9s+3$. If s is odd, then the differences $8s+4, 8s+6, \ldots, 10s$ correspond to the differences of $8s+4, 8s+6, \ldots, 9s+3, v-(9s+5), v-(9s+7), \ldots, v-(10s+2)$, which equal $8s+4, 8s+6, \ldots, 9s+3, 9s+2, 9s, 9s-2, \ldots 8s+5$ (respectively) or, in order, $8s+4, 8s+5, \ldots, 9s+3$. Whether s is even or odd, the differences $8s+4, 8s+5, 8s+6, \ldots, 9s+3$.

Note A.4. Suppose $v \equiv 13 \pmod{18}$ and $v \geq 31$. Say $v = 18s+13$ where $s \geq 1$. Notice that $(3r+2)+(6s-2r+3) = (6s+r+5)$, $(3r+3)+(8s-r+5) = (8s+2r+8)$, $(3r+1)+(4s-r+3) = (4s+2r+4)$, and $(3s+2)+(7s+5) \equiv -(8s+6) \equiv 10s+7 \pmod{18s+13}$ so each of the triples given in the first column of the following table is, in fact, a difference triple. We need to check that all numbers $1, 2, \ldots, (v-1)/2$ (or their negatives modulo v; see Note 1.7.A) are present. Here $(v-1)/2 = 9s+6$.

$\{3r+2, 6s-2r+3, 6s+r+5\}$ for $0 \leq r \leq s-1$	$2, 5, \ldots, 3s-1; 2 \pmod 3$ $4s+5, 4s+7, \ldots, 6s+3$ odd $6s+5, 6s+6, \ldots, 7s+4$
$\{3r+3, 8s-r+5, 8s+2r+8\}$ for $0 \leq r \leq s-1$	$3, 6, \ldots, 3s; 0 \pmod 3$ $7s+6, 7s+7, \ldots, 8s+5$ $8s+8, 8s+10, \ldots, 10s+6$ even$*$
$\{3r+1, 4s-r+3, 4s+2r+4\}$ for $0 \leq r \leq s$	$1, 4, \ldots, 3s+1; 1 \pmod 3$ $3s+3, 3s+4, \ldots, 4s+3$ $4s+4, 4s+6, \ldots, 6s+4$ even
$\{3s+2, 7s+5, 8s+6\}$	$3s+2, 7s+5, 8s+6$

*Notice that all differences of Heffter's Difference Problems are between 1 and

$(v-1)/2 = 9s+6$, so any difference larger than $9s+6$ can be replaced with its negative (mod v), as stated in Note 1.7.A. Hence, if s is even, then the differences $8s+8, 8s+10, \ldots, 10s+6$ correspond to the differences of $8s+8, 8s+10, \ldots, 9s+6, v-(9s+8), v-(9s+10), \ldots, v-(10s+6)$, which equal $8s+8, 8s+10, \ldots, 9s+6, 9s+5, 9s+3, \ldots 8s+7$ (respectively) or, in order, $8s+7, 8s+8, 8s+9, \ldots, 9s+6$. If s is odd, then the differences $8s+8, 8s+10, \ldots, 10s+6$ correspond to the differences of $8s+8, 8s+10, \ldots, 9s+5, v-(9s+7), v-(9s+9), \ldots, v-(10s+6)$, which equal $8s+8, 8s+10, \ldots, 9s+5, 9s+6, 9s+4, 9s+2, \ldots 8s+7$ (respectively) or, in order, $8s+7, 8s+8, \ldots, 9s+6$. Whether s is even or odd, the differences $8s+7, 8s+8, \ldots, 9s+6$.

Note A.5. Suppose $v \equiv 3 \pmod{18}$ and $v \geq 21$. Say $v = 18s+3$ where $s \geq 1$. Notice that $(3r+1)+(8s-r+1) = (8s+2r+2)$, $(3r+2)+(4s-r) = (4s+2r+2)$, and $(3r+3)+(6s-2r-1) = (6s+r+2)$ so each of the triples given in the first column of the following table is, in fact, a difference triple. We need to check that all numbers $1, 2, \ldots, (v-1)/2$ (or their negatives modulo v; see Note 1.7.A) are present. Here $(v-1)/2 = 9s+1$.

$\{3r+1, 8s-r+1, 8s+2r+2\}$ for $0 \le r \le s-1$	$1, 4, \ldots, 3s-2; 1 \pmod 3$ $7s+2, 7s+3, \ldots, 8s+1$ $8s+2, 8s+4, \ldots, 10s$ even$*$
$\{3r+2, 4s-r, 4s+2r+2\}$ for $0 \le r \le s-1$	$2, 5, \ldots, 3s-1; 2 \pmod 3$ $3s+1, 3s+2, \ldots, 4s$ $4s+2, 4s+4, \ldots, 6s$ even
$\{3r+3, 6s-2r-1, 6s+r+2\}$ for $0 \le r \le s-1$	$3, 6, \ldots, 3s; 0 \pmod 3$ $4s+1, 4s+3, \ldots, 6s-1$ odd $6s+2, 6s+3, \ldots, 7s+1$
$\{6s+1\}$	$6s+1$

*Notice that all differences of Heffter's Difference Problems are between 1 and $(v-1)/2 = 9s+1$, so any difference larger than $9s+1$ can be replaced with its negative (mod v), as stated in Note 1.7.A. Hence, if s is even, then the differences $8s+2, 8s+4, \ldots, 10s$ correspond to the differences of $8s+2, 8s+4, \ldots, 9s, v-(9s+2), v-(9s+4), \ldots, v-(10s)$, which equal $8s+2, 8s+4, \ldots, 9s, 9s+1, 9s-1, 9s-3 \ldots 8s+3$ (respectively) or, in order, $8s+2, 8s+3, \ldots, 9s+1$. If s is odd, then the differences $8s+2, 8s+4, \ldots, 10s$ correspond to the differences of $8s+2, 8s+4, \ldots, 9s+1, v-(9s+3), v-(9s+5), \ldots, v-(10s)$, which equal $8s+2, 8s+4, \ldots, 9s+1, 9s, 9s-2, 9s-4, \ldots 8s+3$ (respectively) or, in order, $8s+2, 8s+3, \ldots, 9s+1$. Whether s is even or odd, the differences $8s+2, 8s+3, \ldots, 9s+1$.

Note A.6. Suppose $v \equiv 9 \pmod{18}$ and $v \ge 81$. Say $v = 18s+9$ where $s \ge 4$. Notice that $(3r+1)+(4s-r+3) = (4s+2r+4)$, $(3r+2)+(8s-r+2) =$

$(8s+2r+4)$, and $(3r+3)+(6s-2r+1) = (6s+r+4)$, $(2)+(8s+3) = (8s+5)$, $(3)+(8s+1) = (8s+4)$, $(5)+(8s+2) = (8s+7)$, $(3s-1)+(3s+2) = (6s+1)$, and $(3s)+(7s+3) \equiv -(8s+6) \equiv 10s+3 \pmod{18s+9}$, so each of the triples given in the first column of the following table is, in fact, a difference triple. We need to check that all numbers $1, 2, \ldots, (v-1)/2$ (or their negatives modulo v; see Note 1.7.A) are present. Here $(v-1)/2 = 9s+4$.

$\{3r+1, 4s-r+3, 4s+2r+4\}$ for $0 \le r \le s$	$1, 4, \ldots, 3s+1; 1 \pmod 3$ $3s+3, 3s+4, \ldots, 4s+3$ $4s+4, 4s+6, \ldots, 6s+4$ even
$\{3r+2, 8s-r+2, 8s+2r+4\}$ for $2 \le r \le s-2$	$8, 11, \ldots, 3s-4; 2 \pmod 3$ $7s+4, 7s+5, \ldots, 8s$ $8s+8, 8s+10, \ldots, 10s$ even*
$\{3r+3, 6s-2r+1, 6s+r+4\}$ for $1 \le r \le s-2$	$6, 9, \ldots, 3s-3; 0 \pmod 3$ $4s+5, 4s+7, \ldots, 6s-1$ odd $6s+5, 6s+6, \ldots, 7s+2$
$\{2, 8s+3, 8s+5\}, \{3, 8s+1, 8s+4\}$	$2, 8s+3, 8s+5, 3, 8s+1, 8s+4$
$\{5, 8s+2, 8s+7\}, \{3s-1, 3s+2, 6s+1\}$	$5, 8s+2, 8s+7, 3s-1, 3s+2, 6s+1$
$\{3s, 7s+3, 8s+6\}, \{6s+3\}$	$3s, 7s+3, 8s+6, 6s+3$

*Notice that all differences of Heffter's Difference Problems are between 1 and $(v-1)/2 = 9s+4$, so any difference larger than $9s+4$ can be replaced with its negative (mod v), as stated in Note 1.7.A. Hence, if s is even, then the differences $8s+8, 8s+10, \ldots, 10s$ correspond to the differences of $8s+8, 8s+10, \ldots, 9s+4, v-(9s+6), v-(9s+8), \ldots, v-(10s)$, which equal $8s+8, 8s+10, \ldots, 9s+4, 9s+3, 9s+1, 9s-1 \ldots 8s+9$ (respectively) or, in order, $8s+8, 8s+9, \ldots, 9s+4$. If

s is odd, then the differences $8s+8, 8s+10, \ldots, 10s$ correspond to the differences of $8s+8, 8s+10, \ldots, 9s+3, v-(9s+5), v-(9s+7), \ldots, v-(10s)$, which equal $8s+8, 8s+10, \ldots, 9s+3, 9s+4, 9s+2, 9s, \ldots 8s+9$ (respectively) or, in order, $8s+8, 8s+9, \ldots, 9s+4$. Whether s is even or odd, the differences $8s+8, 8s+9, \ldots, 9s+4$.

Note A.7. Suppose $v \equiv 15 \pmod{18}$ and $v \geq 33$. Say $v = 18s+15$ where $s \geq 1$. Notice that $(3r+1)+(4s-r+3) = (4s+2r+4)$, $(3r+2)+(8s-r+6) = (8s+2r+8)$, and $(3r+3)+(6s-2r+3) = (6s+r+6)$ so each of the triples given in the first column of the following table is, in fact, a difference triple. We need to check that all numbers $1, 2, \ldots, (v-1)/2$ (or their negatives modulo v; see Note 1.7.A) are present. Here $(v-1)/2 = 9s+7$.

$\{3r+1, 4s-r+3, 4s+2r+4\}$ for $0 \leq r \leq s$	$1, 4, \ldots, 3s+1; 1 \pmod 3$ $3s+3, 3s+4, \ldots, 4s+3$ $4s+4, 4s+6, \ldots, 6s+4$ even
$\{3r+2, 8s-r+6, 8s+2r+8\}$ for $0 \leq r \leq s$	$2, 5, \ldots, 3s+2; 2 \pmod 3$ $7s+6, 7s+7, \ldots, 8s+6$ $8s+8, 8s+10, \ldots, 10s+8$ even$*$
$\{3r+3, 6s-2r+3, 6s+r+6\}$ for $0 \leq r \leq s-1$	$3, 6, \ldots, 3s; 0 \pmod 3$ $4s+5, 4s+7, \ldots, 6s+3$ odd $6s+6, 6s+7, \ldots, 7s+5$
$\{6s+5\}$	$6s+5$

*Notice that all differences of Heffter's Difference Problems are between 1 and $(v-1)/2 = 9s+7$, so any difference larger than $9s+7$ can be replaced with its negative (mod v), as stated in Note 1.7.A. Hence, if s is even, then the differences

$8s+8, 8s+10, \ldots, 10s+8$ correspond to the differences of $8s+8, 8s+10, \ldots, 9s+6, v-(9s+8), v-(9s+10), \ldots, v-(10s+8)$, which equal $8s+8, 8s+10, \ldots, 9s+6, 9s+7, 9s+5, 9s+3 \ldots 8s+7$ (respectively) or, in order, $8s+7, 8s+8, \ldots, 9s+7$. If s is odd, then the differences $8s+8, 8s+10, \ldots, 10s+8$ correspond to the differences of $8s+8, 8s+10, \ldots, 9s+7, v-(9s+9), v-(9s+11), \ldots, v-(10s+8)$, which equal $8s+8, 8s+10, \ldots, 9s+7, 9s+6, 9s+4, \ldots 8s+7$ (respectively) or, in order, $8s+7, 8s+8, \ldots, 9s+7$. Whether s is even or odd, the differences $8s+7, 8s+8, \ldots, 9s+7$.

Note. We now have a solution to Heffter's First Difference Problem, which applies to $v \equiv 1 \pmod{6}$, given in Note A.1 (for $v=7$, $v=13$, and $v=19$), Note A.2 (for $v \equiv 1 \pmod{18}$ and $v \geq 37$), Note A.3 (for $v \equiv 7 \pmod{18}$ and $v \geq 25$), and Note A.4 (for $v \equiv 13 \pmod{18}$ and $v \geq 31$). We have a solution to Heffter's Second Difference Problem, which applied to $v \equiv 3 \pmod{6}$ (and $v \neq 9$), given in Note A.1 (for $v=15$, $v=27$, $v=45$, and $v=63$), Note A.5 (for $v \equiv 3 \pmod{18}$ and $v \geq 21$), Note A.6 (for $v \equiv 9 \pmod{18}$ and $v \geq 81$), and Note A.7 (for $v \equiv 15 \pmod{18}$ and $v \geq 33$). As we'll see in the proof of the the necessary and sufficient conditions for the existence of a cyclic Steiner triple system (in Theorem 1.7.6), all we need to do is convert the difference triples given by Heffter's Difference Problems into base blocks (along with a short orbit block in the case that $v \equiv 3 \pmod{6}$, $v \neq 9$) and then to cycle these around with the appropriate cyclic automorphism. These steps can be followed to give a Steiner triple system of any order v (except $v=9$) simply in terms of parameter v.

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