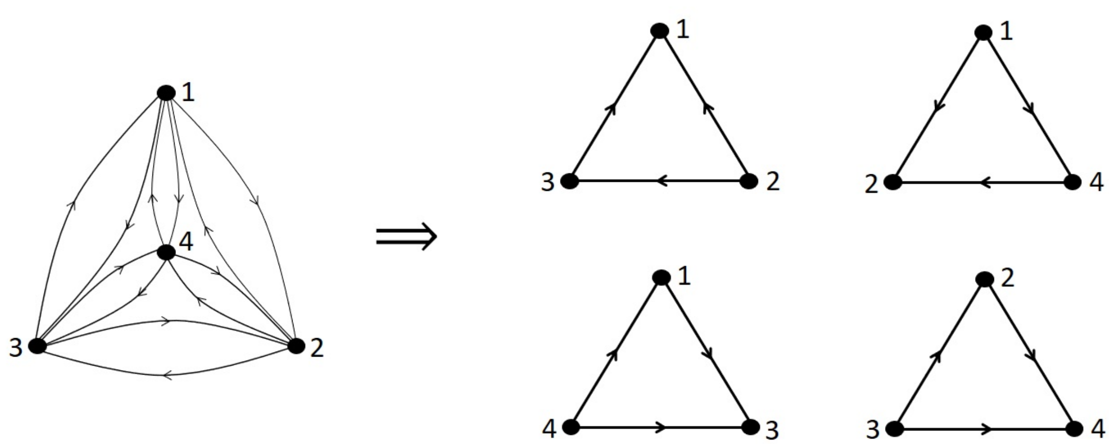
Supplement. Directed and Hybrid Triple Systems

Note. In this supplement we consider arc-disjoint decompositions of the complete digraph on v vertices, D_v, into orientations of a 3-cycle. In Design Theory (not an official ETSU class), Section 2.4. Mendelsohn Triple Systems we considered decompositions into triples of the form $\{(a,b),(b,c),(c,a)\}$ where a,b,c are distinct vertices of a complete digraph. These were called "directed triples" in the Design Theory notes. A better term for this type of triple would be a "Mendelsohn triple." In this supplement, we use the term *directed triple* to indicate a triple of arcs of the form $\{(a,b),(b,c),(a,c)\}$. We denote this triple as $(a,b,c)_D$. These are also sometimes called "transitive triples."

Definition. A directed triple system of order n, denoted $DTS(v)$, is a pair (S,T) where T is an arc disjoint collection of directed triples $\{(a,b),(b,c),(a,c)\}$ which partitions the arc set of the complete digraph D_v on vertex set S.

Example A. A graphical representation of a directed triple system of order 4 is as follows:



Note. Directed triple systems we introduced by Stephen H. Y. Hung and Nathan S. Mendelsohn in "Directed Triple Systems," *Journal of Combinatorial Theory, Series A*, **14**(3), 310-318 (1973). A copy is online on the Science Direct website. It is shown that the spectrum of $DTS(v)$ is all $v \equiv 0$ or 1 (mod 3). As we will show, the construction or Mendelsohn triple systems given in Section 2.4. Mendelsohn Triple Systems can be easily modified to produce directed triple systems.

Note A. For $v \equiv 0 \pmod 3$, $n \neq 6$, we consider $v = 3n$ where $n \neq 2$. Let $(Q, \circ)$ be an idempotent quasigroup of order n and set $S = Q \times \{1, 2, 3\}$. We consider a collection of triples T of two types:

Type 1: The "Type 1" triples $((x,1),(x,2),(x,3))_D$ and $((x,3),(x,2),(x,1))_D$ are in T for each $x \in Q$.

Type 2: The six Type 2 triples $((a,1),(b\circ a,2),(b,1))_D$, $((b,1),(a\circ b,2),(a,1))_D$, $((a,2),(b\circ a,3),(b,2))_D$, $((b,2),(a\circ b,3),(a,2))_D$, $((a,3),(b\circ a,1),(b,3))_D$, $((b,3),(a\circ b,1),(a,3))_D$ belong to T for all $a, b \in Q$ where $a \neq b$.

There are $2n$ Type 1 triples and $6\binom{n}{2}$ Type 2 triples. In Exercise DTS-HTS 1(a), a figure is to be given illustrating these triples.

Note B. For $v \equiv 1 \pmod 3$, $n \neq 7$, we consider $v = 3n + 1$ where $n \neq 2$. Let $(Q, \circ)$ be an idempotent quasigroup of order n and set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. We consider a collection of triples T of two types:

Type 1: The "Type 1" triples $((x,3),(x,1),(x,2))_D$, $(\infty,(x,2),(x,1))_D$, $((x,1),(x,3),\infty)_D$, and $((x,2),\infty,(x,3))_D$ are in T for each $x \in Q$.

Type 2: The six Type 2 triples $((a,1),(b\circ a,2),(b,1))$, $((b,1),(a\circ b,2),(a,1))$, $((a,2),(b\circ a,3),(b,2))$, $((b,2),(a\circ b,3),(a,2))$, $((a,3),(b\circ a,1),(b,3))$, $((b,3),(a\circ b,1),(a,3))$ belong to T for all $a,b\in Q$ where $a\neq b$.

There are $4n$ Type 1 triples and $6\binom{n}{2}$ Type 2 triples. In Exercise DTS-HTS 1(b), a figure is to be given illustrating these triples.

Exercise DTS-HTS 1. **(a)** Draw a figure, similar to the one on pages 57 and 58, illustrating the Type 1 and Type 2 triples of Note A. **(b)** Draw a figure, similar to the one on page 58, illustrating the Type 1 triples of Note B.

Exercise DTS-HTS 2. Show that a $DTS(v)$ contains $v(v-1)/3$ directed triples and therefore a necessary condition for the existence of a $DTS(v)$ is $v\equiv 0$ or 1 (mod 3). (In these exercises and throughout this section we use the term "directed triple" in the sense defined above).

Exercise DTS-HTS 3. Let S be a set of size n and T a collection of directed triples of D_v with vertex set S such that

(a) every arc of D_v belongs to a directed triple of T, and

(b) $|T|\leq v(v-1)/3$.

Prove that (S,T) is an $DTS(v)$. (Compare this to Exercise 2.4.3 in the text book.)

Exercise DTS-HTS 4. Give a $DTS(6)$ and a $DTS(7)$.

Exercise DTS-HTS 5. Prove that the constructions given in Note A and Note B give directed triple systems. Conclude the following theorem.

Theorem DTS-HTS A. The spectrum for directed triple systems, $DTS(v)$, is precisely the set of all $v \equiv 0$ or $1 \pmod 6$.

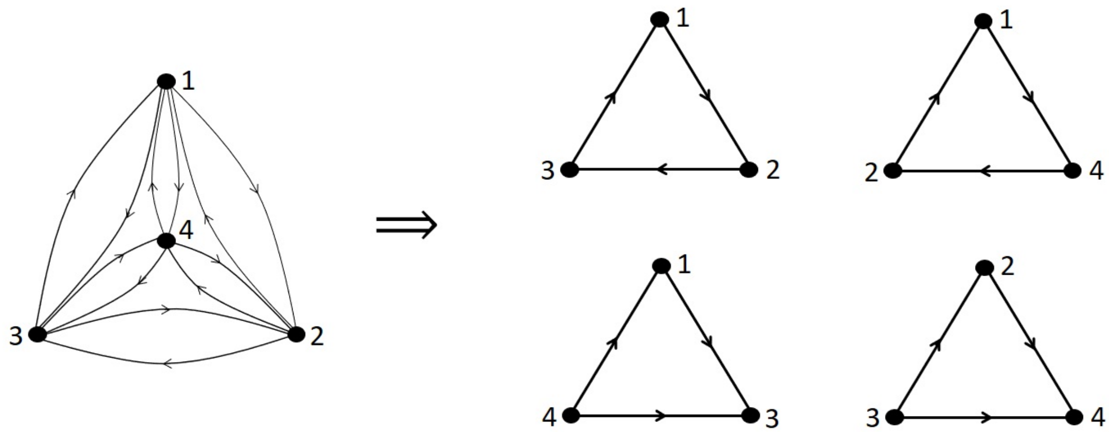
Note. λ-fold directed triple systems were explored by Jennifer Seberry and David Skillcorn in "All Directed BIBDs with $k = 3$ Exist," *Journal of Combinatorial Theory, Series A*, **29**(2), 244–248 (1980). A copy is online on the Science Direct website. The paper includes the following theorem. Of course these are equivalent to arc-disjoint decompositions of λD_v into directed triples.

Theorem DTS-HTS B. A λ-fold directed triple system of order v exists if and only if $\lambda v(v-1) \equiv 0 \pmod 3$, $v \neq 2$.

Note. We now consider arc-disjoint decompositions of the complete digraph D_v into *some* Mendelsohn triples and *some* directed triples. For this discussion, we denote a Mendelsohn triple $\{(a,b),(b,c),(c,a)\}$ as $(a,b,c)_M = (b,c,a)_M = (c,a,b)_M$. We continue to denote the directed triple $\{(a,b),(b,c),(a,c)\}$ as $(a,b,c)_D$.

Definition. An arc-disjoint decomposition of the complete digraph D_v into c Mendelsohn triples and $t = v(v-1)/3 - c$ directed triples is a *c-hybrid triple system* of order v, denoted c-$HTS(v)$. An *oriented triple system* (also called an *ordered triple system*) of order v, denoted $OTS(v)$, is a c-$HTS(v)$ for some c where $0 \leq c \leq v(v-1)/3$.

Example B. A graphical representation of a 2-hybrid triple system of order 4 is as follows:



Note. Notice that hybrid triple systems encompass both Mendelsohn triple systems and directed triple systems; a $MTS(v)$ is a c-$HTS(v)$ where $c = v(v-1)/3$ and a $DTS(v)$ is a 0-$HTS(v)$. Hybrid triple systems were considered in two papers:

1. C. J. Colbourn, W. R. Pulleyblank, and A. Rosa, "Hybrid Triple Systems and Cubic Feedback Sets," *Graphs and Combinatorics*, **5**, 15–28 (1989).

2. Katherine Heinrich, "Simple Direct Constructions for Hybrid Triple Designs," *Discrete Mathematics*, **97**, 223–227 (1991). (This is available online on the Science Direct webpage.

In these papers, the following theorem is established.

Theorem DTS-HTS C. A c-$HTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$, and $c \in \{0, 1, 2, \ldots, v(v-1)/3 - 2, v(v-1)/3\}$.

Note. An internet search seems to indicate that λ-fold hybrid triple systems have not been addresses.

Note. Oriented triple systems were addressed by C. C. Lindner and A. P. Street in "Ordered Triple Systems and Transitive Quasi-Groups," *Ars Combinatoria*, **17A**, 297–306 (1984). The main result of this paper is *not* the existence of oriented triple systems. Notice that Mendelsohn and directed triple systems are examples of oriented triple systems. The necessary and sufficient conditions for the existence of oriented triple systems is given in the following theorem.

Theorem DTS-HTS D. An $OTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3).

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