Supplement. Packings and Coverings for Mendelsohn and Directed Triple Systems

Note. In this supplement we consider packing and covering problems in the setting of Mendelsohn triple systems and directed triple systems. We give definitions and present proofs of some of the claims. The results of this section originally appear in: Robert Gardner, "Optimal Packings and Coverings of the Complete Directed Graph with 3-Circuits and with Transitive Triples," *Congressus Numerantium*, **127**, 161–170 (1997).

Note. Recall that in Supplement. Directed and Hybrid Triple Systems we denoted the triple $\{(a,b),(b,c),(c,a)\}$ containing the arcs $(a,b),(b,c),(c,a)$ as $(a,b,c)_M = (b,c,a)_M = (c,a,b)_M$ and called it a *Mendelsohn triple*. We denoted the directed triple, or "transitive triple," $\{(a,b),(b,c),(a,c)\}$ as $(a,b,c)_D$. See the Figure DPC.1 below. In Section 2.4. Mendelsohn Triple Systems we define a *Mendelsohn triple system* of order v, denoted $MTS(v)$, as a pair (S,T) where T is an arc-disjoint collection of directed triples which partitions the arc set of D_v on vertex set S. In the supplement we defined a *directed triple system* of order v, denoted $DTS(v)$, as a pair (S,T) where T is an arc-disjoint collection of directed triples $\{(a,b),(b,c),(a,c)\}$ which partitions the arc set of the complete digraph D_v on vertex set S. We refer to both Mendelsohn triples and directed/transitive triples as *oriented triangles*. We define packings and coverings in the setting of complete digraphs very similarly to the setting of complete graphs.

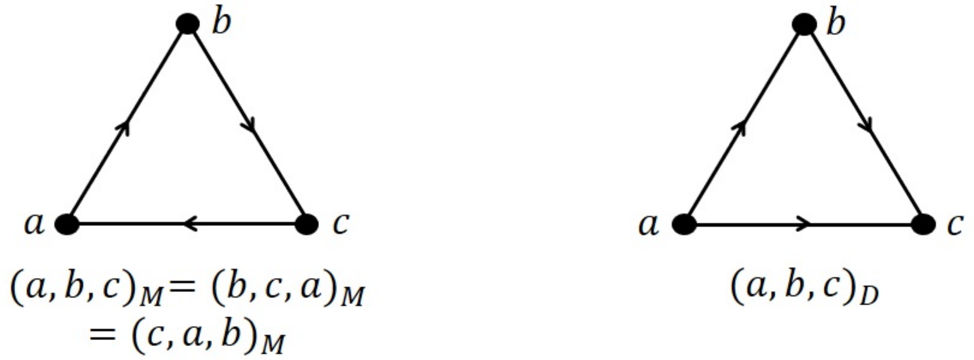

Figure DPC.1. Mendelsohn and directed/transitive triples

Definition. A *packing* of the complete digraph D_v with oriented triangles is a triple (S, T, L), where S is the vertex set of D_v, T is a collection of arc-disjoint oriented triangles from the arc set of D_v, and L is the collection of arcs in D_v not belonging to one of the oriented triangles of T. The collection of arcs L is the *leave*. If $|T|$ is as large as possible, or equivalently if $|L|$ is as small as possible, then (S, T, L) is a *maximum packing* with oriented triangles, or simply a *maximum packing* of order v.

Definition. A *covering* of the complete digraph D_v with oriented triangles is a triple (S, T, P), where S is the vertex set of D_v, P is a subset of the arc set of λD_v based on S (so P may have arcs repeated multiples of times; technically we should speak of the "multiset" P and the arc multiset of λD_v), and T is a collection of arc-disjoint oriented triangles which partition the union of multiset P and the edge set of D_v. The collection of arcs P is the *padding* and v is the *order* of the covering (S, T, P). If $|P|$ is as small as possible then the covering (S, T, P) is a *minimum covering* with oriented triangles, or simply a *minimum covering* of order v.

Note. We consider four problems in this supplement: (1) maximimum packings of order v using directed triples, (2) maximum packings of order v using Mendelsohn triples, (3) minimum coverings of order v using directed triples, and (4) minimum coverings of order v using Mendelsohn triples. Since a $DTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3), then we only need to consider the packing and covering problems in the case $v \equiv 2$ (mod 3). Since a $MTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$, then we only need to consider the packing and covering problems in the cases $v = 6$ and $v \equiv 2$ (mod 3).

Note. We need the following digraphs for the statement of our results.

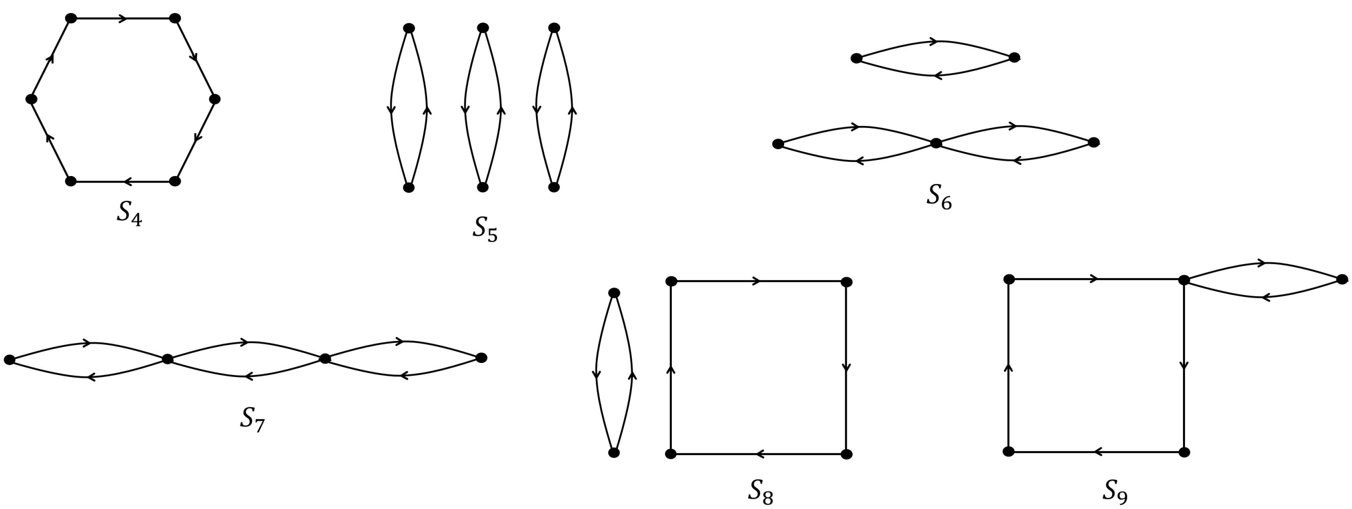

Figure DPC.2. Some leaves and paddings

Note. We now state the four results. We give a partial proof and an example below.

Theorem DPC.A. A maximum packing of D_v with directed/transitive triples satisfies:

1. if $v \equiv 0$ or $1 \pmod 3$ then $L = \varnothing$, and

2. if $v \equiv 2 \pmod 3$ then $L = D_2$.

Note. We now present a proof of Theorem DPC.A in the case $v \equiv 8 \pmod{12}$.

Example DPC.1. With $v = 8$ in the proof of Theorem DPC.A, we have the following 18 directed/transitive triples in a maximal packing of D_8 with leave L where $A(L) = \{(x, y), (y, x)\}$:

$$\{(0, x, 4)_D, (1, x, 5)_D, (2, x, 0)_D, (3, x, 1)_D, (4, x, 2)_D, (5, x, 3)_D\}$$

$$\cup\{(0, y, 5)_D, (1, y, 0)_D, (2, y, 1)_D, (3, y, 2)_D, (4, y, 3)_D, (5, y, 4)_D\}$$

$$\cup\{(0, 2, 3)_D, (1, 3, 4)_D, (2, 4, 5)_D, (3, 5, 0)_D, (4, 0, 1)_D, (5, 1, 2)_D\}.$$

Theorem DPC.B. A maximum packing of D_v with Mendelsohn triples satisfies:

1. if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$, then $L = \varnothing$,

2. if $v = 6$ then $L \in \{S_4, S_5, S_6, S_7, S_8, S_9\}$, and

3. if $v \equiv 2 \pmod 3$ then $L = D_2$.

Theorem DPC.C. A minimum covering of D_v with directed/transitive triples satisfies:

1. if $v \equiv 0$ or $1 \pmod 3$ then $P = \varnothing$, and

2. if $v \equiv 2 \pmod 3$ then P has four arcs and may be two disjoint copies of D_2, any orientation of a 4-cycle, or S_6.

Theorem DPC.D. A minimum covering of D_v with Mendelsohn triples satisfies:

1. if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$, then $P = \varnothing$,

2. if $v = 6$ then $P = C_3$, and

3. if $v \equiv 2 \pmod 3$ then P has four arcs and may be two disjoint copies of D_2, any orientation of a 4-cycle, or two copies of D_2 which share a single vertex.

Note. We now present a proof of Theorem DPC.D in the case $v \equiv 2 \pmod 6$.

Example DPC.2. With $v = 14$ in the proof of Theorem DPC.D (in the case where P is two disjoint copies of D_2), we have the following 62 Mendelsohn triples in a minimal covering of D_{14}:

$$\{(0,2,5)_M, (1,3,6)_M, (2,4,7)_M, (3,5,8)_M, (4,6,0)_M,$$

$$(5,7,1)_M,(6,8,2)_M,(7,0,3)_M,(8,1,4)_M\}$$

$$\{(1,2,a)_M,(2,3,a)_M,(3,4,a)_M,(4,5,a)_M,(5,6,a)_M,$$

$$(6,7,a)_M,(7,8,a)_M,(8,7,a)_M,(0,8,a)_M\}$$

$$\cup\{(0,5,b)_M,(1,6,b)_M,(2,7,b)_M,(3,8,b)_M,(4,0,b)_M,$$

$$(5,1,b)_M,(6,2,b)_M,(7,3,b)_M,(8,4,b)_M\}$$

$$\cup\{(0,6,c)_M,(1,7,c)_M,(2,8,c)_M,(3,0,c)_M,(4,1,c)_M,$$

$$(5,2,c)_M,(6,3,c)_M,(7,4,c)_M,(8,5,c)_M\}$$

$$\cup\{(0,7,d)_M,(1,8,d)_M,(2,0,d)_M,(3,1,d)_M,(4,2,d)_M,$$

$$(5,3,d)_M,(6,4,d)_M,(7,5,d)_M,(8,6,d)_M\}$$

$$\cup\{(0,8,e)_M,(1,0,e)_M,(2,1,e)_M,(3,2,e)_M,(4,3,e)_M,$$

$$(5,4,e)_M,(6,5,e)_M,(7,6,e)_M,(8,7,e)_M\}$$

$$\cup\{(a,b,e)_M,(a,e,b)_M,(a,e,c)_M,(a,d,e)_M,(a,c,d)_M,$$

$$(c,e,d)_M,(b,c,d)_M,(b,d,c)_M\}.$$

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