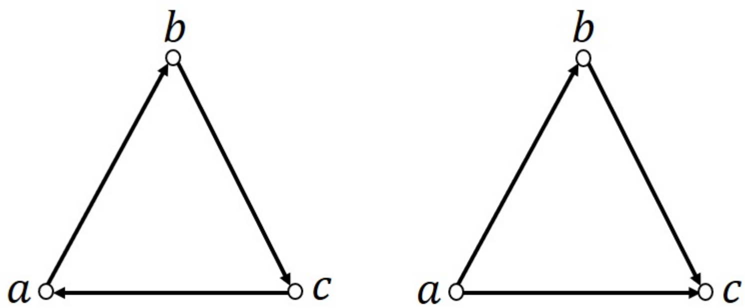
Supplement. Mixed Triple Systems

Note. In this supplement, we consider the complete mixed graph M_n and various isomorphic decompositions into copies of a given small mixed graph on three vertices.

Note. Recall that a *Steiner triple system* of order n is an isomorphic decomposition of $G = K_n$ into a family $\mathcal{F}$ of subgraphs of G such that each $F \in \mathcal{F}$ is isomorphic to a 3-cycle. As we have seen a Steiner triple system of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.

Note. Recall that there are two orientations of the 3-cycle:

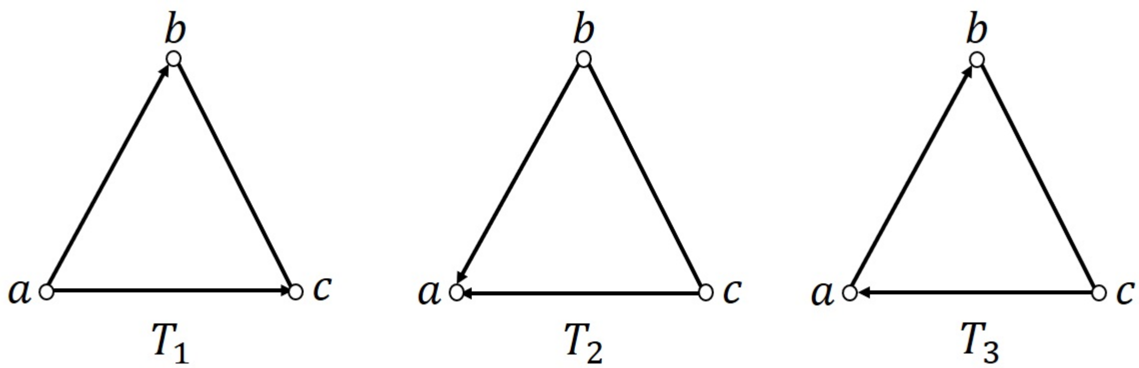


Mendelsohn Triple Directed Triple

With D_n the complete digraph, a decomposition of D_n into Mendelsohn triples is called a *Mendelsohn triple system* or order n, denoted $MTS(n)$. A decomposition of D_n into directed triples is called a *directed triple system* of order n, denoted $DTS(n)$. A $MTS(n)$ exists if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$. A $DTS(n)$ exists if and only if $n \equiv 0$ or $1 \pmod 3$. For references on these results, see Section 2.4. Mendelsohn Triple Systems and Supplement. Directed and Hybrid Triple Systems.

Definition. A *mixed graph* consists of a vertex set, an edge set, and an arc set. The *complete mixed graph* on n vertices, denoted M_n, has for each pair u and v of distinct vertices, an edge joining u and v, an arc from u to v, and an arc from v to u. So M_n has twice as many arcs as edges.

Note. We can extend the ideas of Steiner triple systems, Mendelsohn triple systems, and directed triple systems to mixed graphs. There are three distinct partial orientations of a 3-cycle which have twice as many arcs as edges (as does the complete mixed graph):



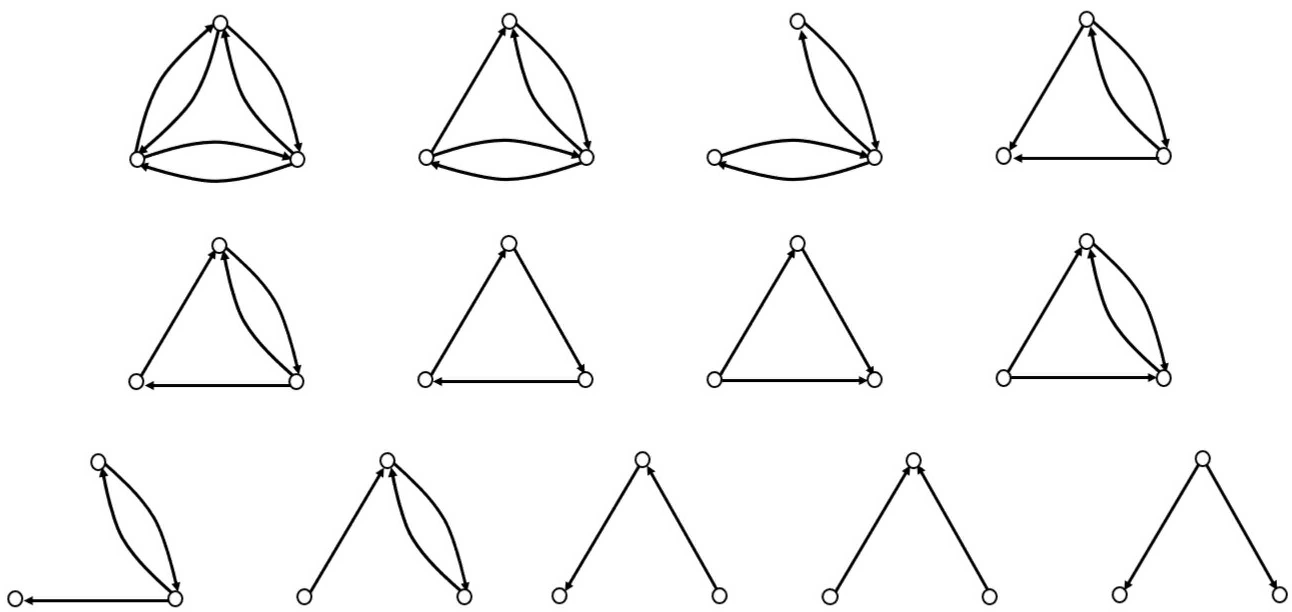
We call these *mixed triples* and denote mixed triple T_i as $[a, b, c]_i$ where $i \in \{1, 2, 3\}$. Mixed triple systems were introduced in 1999 by Robert "Dr. Bob" Gardner and necessary and sufficient conditions were given for their existence [5].

Definition. A decomposition of the complete mixed graph on n vertices into copies of T_i is a T_i*-mixed triple system*, where $i \in \{1, 2, 3\}$.

Theorem M.1. A T_i-mixed triple system of order n exists for each $i \in \{1, 2, 3\}$ if and only if $n \equiv 1 \pmod 2$, except in the cases $n \in \{3, 5\}$ when $i = 3$.

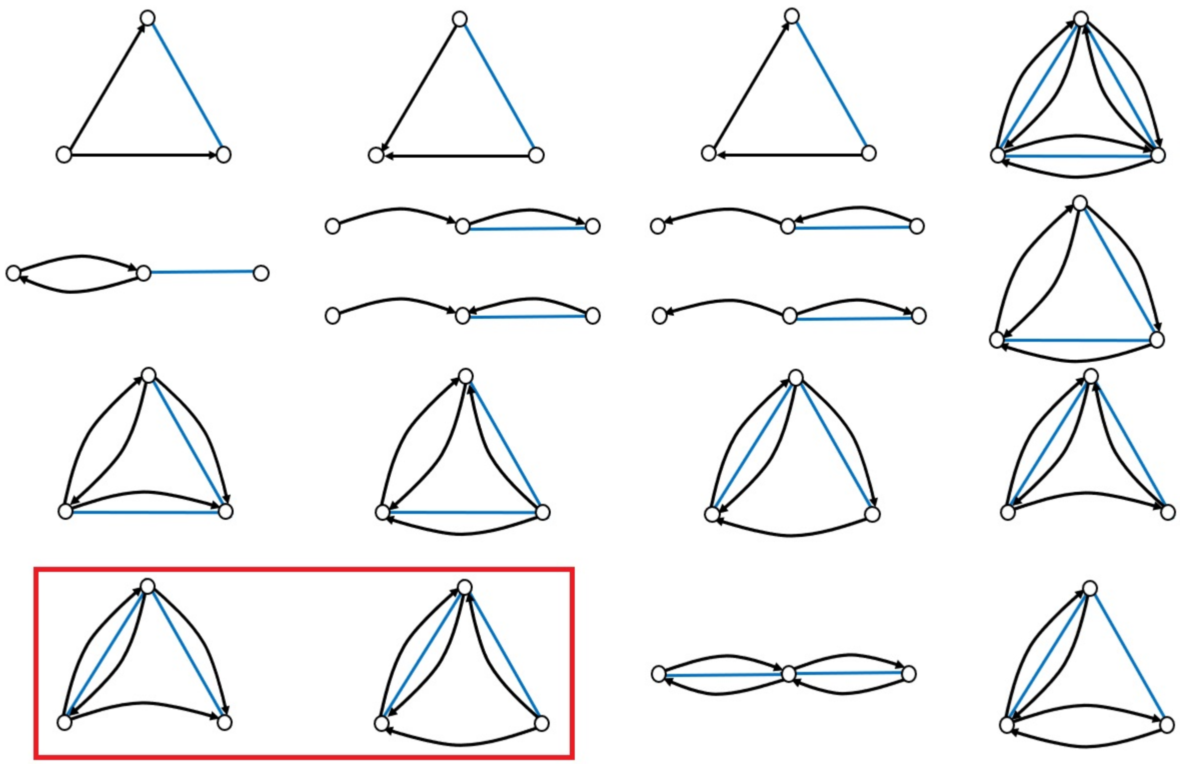
Note. Along the lines of a hybrid triple system in the setting of digraphs (see Supplement. Directed and Hybrid Triple Systems), we could consider hybrid mixed triple systems. This would be a decomposition of the complete mixed graph into m_1 copies of T_1, m_2 copies of T_2, and m_3 copies of T_3 where $m_1+m_2+m_3 = n(n-1)/2$. To date, nothing has been done in this direction (to my knowledge...), making this an **open problem**.

Note. In 1986 Alan Hartman and Eric Mendelsohn considered all (strict) digraphs on three vertices (of which there are 13, up to isomorphism):



They then defined 13 new types of "triple systems" in terms of decompositions of D_n into copies of each of these 13 digraphs. They gave necessary and sufficient conditions for the existence of each such triple system in their paper "The Last of the Triple Systems" [6].

Note. In the spirit of Hartman and Mendelsohn, ETSU graduate student Ernest Jum in 2009 considered all (simple/strict) mixed graphs on three vertices with (like the complete mixed graph) twice as many arcs as edges (of which there are 18, up to isomorphism) in his master's thesis "The Last of the Mixed Triple Systems" [7]:



Mr. Jum gave necessary and sufficient conditions for the existence of a decomposition for the complete mixed graph into each of the triples, with the exception of the two mixed graphs outlined in red here (which are converses of each other), making this an **open problem**.

Note. The packing problem for mixed triples was addressed in part in Benkam Benedict Bobga's 2005 ETSU master's thesis "Bicyclic Mixed Triple Systems" [2]. See his "Chapter 4. The Packing Problem" (the thesis primarily addressed automorphisms of mixed triple systems; the automorphism results were published in

[3]). We will address packing problems (and associated covering problems) in Chpater 4 (Maximum Packings and Minimum Coverings); in particular, see Secion 4.2. Maximum Packings, Section 4.3. Minimum Coverings, and (for similar problems in the digraph case) Supplement. Packings and Coverings for Mendelsohn and Directed Triple Systems. The covering problem and the remainder of the packing problem are **open problems**. It seems that existence of λ-fold mixed triple systems has not been addressed and is an **open problem**.

Note. Since the complete graph M_n has twice as many arcs as edges, we can address isomorphic decompositions of M_n into any mixed graph with twice as many arcs as edges. Such decompositions into copies of partial orientations of paths P_4 and stars S_3 (with twice as many arcs as edges) were given by Beeler and Meadows in 2009 [1]. Decompositions of M_n into copies of partial orientations of the stars S_6 (with twice as many arcs as edges) were given by Culver and Gardner in 2020 [4]. Notice that in such decompositions, the subgraph (or "block") must have twice as many arcs as edges and hence its underlying graph must have a multiple of three edges; this is why stars S_3 and S_6 have been considered). In terms of partial orientations of cycles, the next logical thing to consider after mixed triple systems would be "mixed hexagon systems" which consider partial orientations of a 6-cycle; this appears to be **an open problem**.

Note. Breaking News! The existence of mixed hexagon systems has been resolved! There are 25 partial orientations of a 6-cycle which has two edges and four arcs. These are called *mixed hexagons*. A decomposition of the complete mixed graph M_n into one of the mixed hexagons is called a *mixed hexagon system of order* n. It has been shown by R. Gardner and S. Ignace that for each of the 25 partial hexagons, such a mixed hexagon system of order n exists if and only if $n \equiv 1 \pmod 4$, $n \geq 9$. These results are currently (July 2023) submitted to the *Bulletin of the Institute of Combinatorics and Its Applications* for possible publication. You can view the manuscript online on Dr. Bob's Publications webpage. This opens the door to the **open problems** of the associated λ-fold mixed hexagon systems, and the associated packing and covering problems.

References

[1] Robert A. Beeler and Adam M. Meadows, Decomposition of Mixed Graphs Using Partial Orientations of P_4 and S_3, *International Journal of Pure Applied Mathematics*, **56** (2009) 63-67.

[2] Benkam Benedict Bobga, "Bicyclic Mixed Triple Systems." (2005). East Tennessee State University Electronic Theses and Dissertations. Paper 1043. Available online here.

[3] Benedict Bobga nd Robert Gardner, Bicyclic, Rotational, and Reverse Mixed Triple Systems, *Bulletin of the Institute of Combinatorics and its Applications*, **48** (2006), 45–52.

[4] Chancé Culver and R. Gardner, Decompositions of the Complete Mixed Graph into Mixed Stars, *International Journal of Innovation in Science and Mathematics*, **8**(3) (2020), 110–114.

[5] Robert Gardner, Triple Systems from Mixed Graphs, *Bulletin of the Institute of Combinatorics and its Applications*, **27** (1999), 95–100.

[6] Alan Hartman and Eric Mendelsohn, The Last of the Triple Systems, *Ars Combinatorics* **22** (1986), 25–41.

[7] Ernest Jum, "The Last of the Mixed Triple Systems." (2009). East Tennessee State University Electronic Theses and Dissertations. Paper 1876. Available online here.

Revised: 7/4/2023