Chapter I. Examples and Basic Definitions

1.1. Incidence Structures and Incidence Matrices

Note. In this section we define incidence structures and give some (probably familiar) examples. We define the degree of points and blocks, and "simple" incidence structures. Necessary and sufficient conditions are given for a simple incidence structure with constant point degree and constant block degree (see Theorem 1.9).

Definition 1.1. An *incidence structure* is a triple $\boldsymbol{D} = (V, \boldsymbol{B}, I)$ where V and $\boldsymbol{B}$ are any two disjoint sets and I is a binary relation between V and $\boldsymbol{B}$; that is, $I \subseteq V \times \boldsymbol{B}$. The elements of V are *points*, the elements of $\boldsymbol{B}$ are *blocks*, and elements of I are *flags*. We denote $(p, B) \in I$ as $p\,I\,B$ and state that "the point p lies on the block B," "B padded through p," or "p and B are incident."

Note. We will consistently use lower case letters to represent points and upper case letters to represent blocks. As a familiar example, we could take $V = \mathbb{R}^n$, $\boldsymbol{B}$ as the set of all lines in $\mathbb{R}^n$ (that is, the set of all translations of 1-dimensional subspaces), and define the binary relation I as $p\,I\,B$ means that point $p \in \mathbb{R}^n$ is an element of line B. **In this book (and these notes), we almost exclusively deal with finite incidence structures!** We are interested in classes of incidence structures that are somewhat more specific that those given by Definition 1.1.

Definition. Let $\boldsymbol{D} = (V, \boldsymbol{B}, I)$ be an incidence structure. For point $p \in V$, define the set $(p) = \{B \in \boldsymbol{B} \mid p\, I\, B\}$. In $Q \subseteq V$ then define the set $(Q) = \{b \in \boldsymbol{B} \mid p\, I\, B \text{ for each } p \in Q\}$. Since we only consider finite incidence structures, then we can list the elements of these sets and we use the notation $(p) = (p_1, p_2, \ldots, p_m)$, for example. For $\boldsymbol{C} \subseteq \boldsymbol{B}$ we define the set $(\boldsymbol{C}) = \{p \in V \mid p\, I\, B \text{ for each } B \in \boldsymbol{C}\}$. The *degree* or point p is the cardinality $|(p)|$ (the number of blocks on which point p lies). The *degree* of block B is the cardinality $|\{p \in V \mid p\, I\, B\}|$ (the number of points that lie on block B).

Note. We have not disallowed two distinct blocks containing exactly the same points. That is, we may have $(B_1) = (B_2)$ for distinct blocks B_1 and B_2. If this is the case, then we say that these blocks are "repeated" and will want to state their multiplicities. Often, this case will be avoided and motivates the next definition.

Definition 1.2. An incidence structure is *simple* if $(B) \neq (C)$ whenever B and C are distinct blocks. The *trace* of block B is the set $(B) = \{x \in V \mid x\, I\, B\}$. The *degree* of block B is the cardinality $|(B)| = |\{x \in V \mid x\, I\, B\}|$ (the number of points that lie on block B).

Example 1.3. Let $V = \{0, 1, 2, 3, 4, 5, 6\}$ and $\boldsymbol{B} = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$. Take as the incidence relation I the membership relation $\in$. This is an example of a simple incidence structure. Notice that each block is of degree three and each point is of degree three. The structure is, of course, a Steiner triple system of order seven. See my online notes for senior/graduate level Design Theory (not a formal ETSU class) on Section 1.1. The Existence Problem; notice Example 1.1.1(c) and Note 1.1.A. A drawing of this incidence structure is given in Figure 1.1, with the obvious meaning of the blocks and incidence relation.

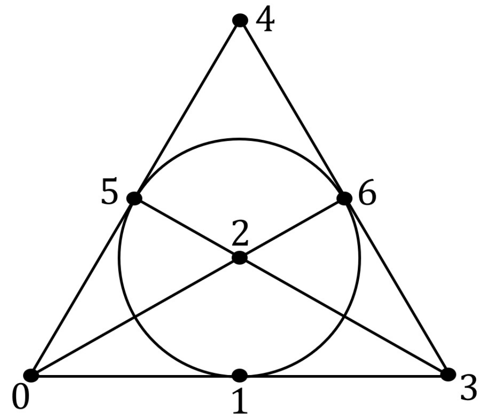

Figure 1.1. A visual representation of the incidence structure of Example 1.3.

Note. Another convenient representation of an incidence structure is the incidence matrix. Later, we will use linear algebraic techniques to to explore the incidence structure based on properties of the incidence matrix.

Definition 1.4. Let $\boldsymbol{D} = (V, \boldsymbol{B}, I)$ be a finite incidence structure and label the points $p_1, p_2, \ldots, p_v$ and the blocks $B_1, B_2, \ldots, B_b$. Then the matrix $M = (m_{ij})$, where $i \in \{1, 2, \ldots, v$ and $j \in \{1, 2, \ldots, b\}$, is

$$m_{ij} = \begin{cases} 1 & \text{if } p_i\, I\, B_j \\ 0 & \text{otherwise} \end{cases}$$

is an *incidence matrix* of $\boldsymbol{D}$. The column of M belonging to block b is the *incidence vector* of B.

Note. Notice that the incidence matrix depends on the labeling of the points and blocks. If the labels are permuted in some way, then the incidence matrix can be correspondingly permuted. In fact, any $m \times n$ matrix of 0's and 1's determines an incidence structure on m points and n blocks.

Note. It is straightforward to see that the seven points and seven blocks of the Steiner triple system of Example 1.3 has incidence matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Note. Notice that each point degree and each block degree is 3 in the Steiner triple system of Example 1.1; this is easily seen in Example 1.3, since each row sum and column sum is 3. We might want to find an incidence structure with *given* point degrees and block degrees. Necessary and sufficient conditions are known for this (the book references Jungnickel's *Graphs, Networks and Algorithms*, Springer (1999) as given a proof). Next, we state an easy necessary (though not in general sufficient) condition for the existence of such an incidence structure.

Proposition 1.6. A necessary condition for the existence of an incidence structure with point degrees $r_1, r_2, \ldots, t_v$ and block degrees $k_1, k_2, \ldots, k_b$ is:

$$\sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j.$$

Corollary 1.7. Let $\boldsymbol{D}$ be an incidence structure with v points, b blocks, all points with degree r, and all blocks with degrees k. Then $vr = bk$.

Note. For our first nontrivial result, we will prove give necessary and sufficient conditions for the existence of a simple design with all points of constant degree r and all blocks of constant degree k. In Chapter 3, such structures will be called "simple 1-designs." The construction we give is due to David Billington and appears in "A Simple Proof that all 1-Designs Exist," *Discrete Mathematics*, **42**, 321–322 (1982). The original paper can be viewed on the ScienceDirect.com webpage. We need a preliminary lemma.

Lemma 1.8. Suppose the existence of a simple incidence structure $\boldsymbol{D}$ on v points with b blocks, constant block degrees k and point degrees $r_1, r_2, \ldots, r_v$. If one has $r_i > r_j$ for a pair of indices i, j (with $i > j$, say), then there also exists a simple incidence structure $\boldsymbol{D}'$ on v points with b blocks, constant block degrees k and point degrees $r_1, r_2, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{j-1}, r_j + 1, r_{j+1}, \ldots, r_v$.

Theorem 1.9. A simple incidence structure on v points with b blocks, constant block degrees k and constant point degrees r exists if and only if

$$vr = bk \text{ and } b \leq \binom{v}{k}.$$

Note. The terminology we use in incidence structures is suggestive of the terminology from graph theory. Recall that the complement of a (simple) graph G is the graph H with the same vertex set as G, and with vertices u and v adjacent in H if and only if u and v are not adjacent in G. The complementary nature of G and H is that their two edge sets partition the edge set of a complete graph on the common vertex set. We take this as motivation for the following definition.

Definition 1.10. The *complementary structure* of an incidence structure $\boldsymbol{D} = (V, \boldsymbol{B}, I)$ is the incidence structure $\overline{\boldsymbol{D}} = (V, \boldsymbol{B}, J)$ with $J = (V \times \boldsymbol{B}) \setminus I$. That is, $x\, J\, B$ if and only if $x \not\!I\, B$ for all $x \in V$ and for all $B \in \boldsymbol{B}$ (here, "$x \not\!I\, B$" indicates that x does not belong to B in $\boldsymbol{B}$).

Note. The incidence matrix for $\overline{\boldsymbol{D}}$ can be obtained from the incidence matrix M for $\boldsymbol{D}$ by changing the 0's of M to 1's, and changing the 1's of M to 0's.

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