1.2. Block Designs and Examples From Affine and Projective Geometry

Note. In this section we define a finite projective plane, state some quantitative properties, and show that there exists a projective plane of all orders a power of a prime (in Proposition 2.3). We define an affine plane and state some quantitative properties. We show that a projective plane of order n can be used to construct an affine plane of order n, and conversely (in Proposition 2.7). Mush of this material is also covered in the senior/graduate level class Design Theory (see Chapter 7, "Affine and Projective Planes"). We define the more general n-dimensional projective space, and define a block design in great generality along with some necessary conditions for their existence. As a special case, we consider pairwise balanced designs.

Definition 2.1. An incidence structure $\boldsymbol{D} = (V\boldsymbol{B}, I)$ is a *projective plane* if and only if it satisfies the following axioms:

(2.1.a) Any two distinct points are joined by exactly one line (or "block").

(2.1.b) Any two distinct lines intersect in a unique point.

(2.1.c) There exists a *quadrangle*, i.e. four points no three of which are on a common line.

Note. The incidence structure of Example 1.3 and Figure 1.1 is an example of a projective plane (in fact, this is the projective plane with the least number of points). The next result is seen in undergraduate/graduate Design Theory in Section 7.2. Projective Planes (see Exercise 7.2.2). Recall that for $p \in V$, $|(p)|$ is the number of lines on which point p lies, and for $G \in \boldsymbol{B}$, $|(G)|$ is the number of points on line G, $|V|$ is the total number of points, and $|\boldsymbol{B}|$ is the total number of lines.

Proposition 2.2. Let $\boldsymbol{D} = (V, \boldsymbol{B}, I)$ be a finite projective plane. Then there exists a natural number n, called the *order* of $\boldsymbol{D}$, satisfying:

(2.2.a) $|(p)| = |(G)| = n + 1$ for all $p \in V$ and $G \in \boldsymbol{B}$;

(2.2.b) $|V| = |\boldsymbol{B}| = n^2 + n + 1$.

Note. Notice that we cannot have a projective plane with $n = 1$, since this would imply that there are only 3 vertices in violation to (2.1.c) of Definition 2.1. With $n = 2$, we have 7 vertices, 7 lines, 3 points on each line, and each point lying on 3 lines; this is the projective plane of Example 1.3 and Figure 1.1.

Note. Next, we construct projective planes of all orders of a power of a prime, $q = p^k$ for some $k \in \mathbb{N}$. The construction is based on finite fields of order q. Such fields are called *Galois fields*. The proof that such structures exist is given in Introduction to Modern Algebra 2 (MATH 4137/5137) in Section VI.33. Finite Fields. The next result is addressed in undergraduate/graduate Design Theory in Section 7.2. Projective Planes; see Exercise 7.2.2.

Proposition 2.3. For each prime power q, there exists a projective plane of order q.

Note. The projective planes constructed in the proof of Proposition 2.3 are denoted $PG(2, q)$. These are not the only projective planes. In the next section we'll see a way to characterize whether a projective plane arises from the construction of Proposition 2.3 or not (see Example 3.6).

Definition 2.4. An incidence structure $\boldsymbol{D} = (V, \boldsymbol{B}, I)$ is an *affine plane* if and only if it satisfies the following axioms:

(2.4.a) Any two distinct points are joined by exactly one line.

(2.4.b) Given any point p and any line G with $p \not{I} G$ there is precisely one line H with $p\, I\, H$ and not intersection G.

(2.4.c) There is a *triangle*, i.e. three points not on a common line.

Two lines G and H are *parallel* if $G = H$ or $|(G, H)| = 0$, denoted $G \parallel H$.

Note. We can reword (2.4.b) of Definition 2.4 as: "Given any line G and a point p not on the line, there is exactly on line H containing point p parallel to line G." This is the Parallel Postulate from Euclidean geometry; actually, it is Playfair's Theorem (an equivalent to the Parallel Postulate). For more details, see my online presentation on Euclidean Geometry.

Proposition 2.5. Let $\boldsymbol{D} = (V, \boldsymbol{B}, I)$ be an affine plane. Then parallelism is an equivalence relation on $\boldsymbol{B}$. If $\boldsymbol{D}$ is finite, then there exists a natural number n (called the *order* of $\boldsymbol{D}$) satisfying:

(2.5.a) $|(p)| = n + 1$ for all points p;

(2.5.b) $|(G)| = n$ for all lines G;

(2.5.c) $|V| = n^2$ and $|\boldsymbol{B}| = n^2 + n$.

Note.

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