Chapter 10. Partial Differential Equations and Fourier Series

Section 10.1. Separation of Variables; Heat Conduction

Note. In the area of PDEs, there are three basic equations which are studied:

1. the heat equation,

2. the wave equation, and

3. the potential equation.

Fourier series are often used in the solution of these equations. We will derive the heat equation and try to give some idea of why Fourier series arise.

Note. The following discussion of the heat equation is based on Appendix A (page 572). Suppose two parallel plates are separated by a distance d. Suppose the plates have temperatures T_1 and T_2. Then according to Newton's Law of Cooling, the amount of heat transferred from the warmer to the colder plate is proportional to the area of the plates (say A), the temperature difference $|T_2 - T_1|$, and inversely proportional to d:

$$\text{amount of heat per unit time} = \kappa A|T_2 - T_1|/d$$

where κ is called the *thermal conductivity* of the medium between the plates.

Note. We want to consider the distribution of heat in a thin rod (this will give us the "One-Dimensional Heat Equation"). Suppose we have such a rod and it lies along the x-axis. Also suppose that the rod is perfectly insulated around the sides (no heat passes through them). So the temperature of the rod can be considered as a function of x and time t. Say, temp $= u = u(x, y)$. Also suppose the ends of the rod are at $x = 0$ and $x = \ell$.

Note. Consider some point $x = x_0$ in the rod. By Newton's Law of Cooling, the heat flow through this point (or heat flux) is

$$H(x_0, t) = -\lim_{d \to 0} \kappa A \frac{u(x_0 + d/2, t) - u(x_0 - d/2, t)}{d} = -\kappa A \left.\frac{\partial u}{\partial x}\right|_{(x_0,t)} = -\kappa A u_x(x_0, t).$$

Similarly, at the point $x = x_0 + \Delta x$, the heat flux is

$$H(x_0 + \Delta x, t) = -\kappa A u_x(x_0 + \Delta x, t).$$

So in the segment of the rod from x_0 to $x_0 + \Delta x$ the net heat flow rate is

$$Q = H(x_0, t) - H(x_0 + \Delta x, t) = -\kappa A \left(u_x(x_o, t) - u_x(x_0 + \Delta x, t)\right)$$

and the amount of heat entering in time Δt is

$$Q\Delta t = \kappa A \left(u_x(x_0 + \Delta x, t) - u_x(x_0, t)\right) \Delta t. \qquad (*)$$

Now, the change in temperature Δu in the time interval Δt is proportional to the amount of heat $Q\Delta t$ and inversely proportional to the mass Δm:

$$\Delta u = \frac{1}{s}\frac{Q\Delta t}{\Delta m} = \frac{1}{s}\frac{Q\Delta t}{\rho A \Delta x}$$

where the constant of proportionality s is called the *specific heat* of the rod and ρ is the density.

Note. The change in temperature Δu of the Δx segment must equal the temperature change at some point in the segment (by the Mean Value Theorem). Say at $x = x_0 + \theta\Delta x$ where $\theta \in (0,1)$:

$$\Delta u - u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t) = \frac{Q\Delta t}{s\rho A\Delta x}$$

or

$$A\Delta t = (u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t))\, s\rho A\Delta x. \qquad (**)$$

From $(*)$ and $(**)$

$$\kappa A\,(u_x(x_0 + \Delta x, t) - u(x_0, t))\,\Delta t = s\rho A\,(u)x_0 + \theta\Delta x, t + \Delta t) - u(c_0 + \theta x, t))\,\Delta x.$$

Dividing by $\Delta x\Delta t$:

$$\kappa A\frac{u_x(x_0 + \Delta x, t) - u(x_0, t)}{\Delta x} = s\rho A\frac{u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t)}{\Delta t}.$$

Now letting $\Delta x \to 0$ and $\Delta t \to 0$ we get

$$\kappa A u_{xx}(x_0, t) = s\rho A u_t(x_0, t)$$

or

$$\frac{\kappa}{s\rho}u_{xx}(x_0, t) = \alpha^2 u_{xx}(x_0, t) = u_t(x_0, t).$$

This is valid for each point $x_0 \in (0, \ell)$ so we want

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \text{ for } x \in (0, \ell), t > 0.$$

This is the one dimensional *heat equation*.

Note. We now want to solve the heat equation given certain initial conditions and boundary conditions. Suppose the temperature distribution is initially given by the function $f(x)$ for $x \in [0, \ell]$. Also suppose the ends of the rod heave constant temperatures (we will use a temperature of 0 for both ends; this is without a loss of generality, as we will see). So we have:

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \text{ for } x \in (0, \ell), t > 0 \\ \text{IC: } u(x, 0) = f(x) \text{ for } x \in [0, \ell] \\ \text{BC: } u(0, t(= 0, u(\ell, t) = 0 \text{ for } t > 0. \end{cases}$$

We will apply the method of separation of variables to solve this. We seek solutions of the form $u(x, t) = X(x)T(t)$. This PDE becomes $\alpha^2 X''T = XT'$ where primes represent ordinary derivatives. This rearranges to become

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}.$$

Since the left hand side is a function of x alone and the right hand side is a function of t above, in order to be equal, both sides must simply be constant. So let

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\sigma.$$

We get

$$\begin{cases} X'' + \sigma X = 0 \\ T' + \alpha^2 \sigma T = 0. \end{cases}$$

Now, solutions to these DEs yield the desired function $u(x, t)$: $u(x, t) = X(x)T(t)$. We need $u(0, t) = X(0)T(t) = 0$. So we need either $T(t) = 0$ (which means $u(x, t) = 0$, which is bad!) or $X(0) = 0$. Similarly we need $X(\ell) = 0$. So we are lead to the two-point boundary value problem:

$$\begin{cases} X'' + \sigma X = 0 \\ X(0) = 0 \\ X(\ell) = 0. \end{cases}$$

With $\sigma \leq 0$ this only has the trivial solution $X(x) = 0$ (see page 515). So, suppose $\sigma = \lambda^2 > 0$. The general solution to $X'' + \lambda^2 X = 0$ is

$$X(x) = k_1 \cos \lambda x + k_2 \sin \lambda x.$$

So if we require $X(0) = 0$, we may take $k_1 = 0$. With $X(\ell) = 0$, we need $X(\ell) = k_2 \sin \lambda\ell = 0$. We don't want $k_2 = 0$ (this would give $X(x) = 0$), so we choose $\lambda\ell n\pi$ where $n = 1, 2, 3, \ldots$. That is, $\lambda = n\pi/\ell$ and so $\sigma = n^2\pi^2/\ell^2$ where $n = 1, 2, 3, \ldots$. So we have $X(x) = k_3 \sin(n\pi x/\ell)$. Returning now to T:

$$T' + \alpha^2 \sigma T = 0 \text{ or } T' + \frac{n^2\pi^2\alpha^2}{\ell^2} T = 0$$

which gives

$$T(t) = k_4 e^{-n^2\pi^2\alpha^2 t/\ell^2}.$$

So we have the following candidates for u:

$$u_n(x,t) = k_n e^{-n^2\pi^2\alpha^2 t/\ell^2} \sin(n\pi x/\ell) \text{ for } n = 1, 2, 3, \ldots.$$

Each of the u_n satisfies the PDE and the boundary conditions. In fact, linear combinations of the u_n's satisfy these (the PDE is linear and, from the boundary conditions, homogeneous). Therefore, we let

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/\ell^2} \sin(n\pi x/\ell).$$

The initial condition then becomes:

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/\ell) = f(x).$$

So finding a solution to the PDE with BC and IC becomes the problem of expressing the function $f(x)$ in terms of an infinite sine series.

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