Sections 10.2 and 10.5. Fourier Series (Partial)

Note. We now consider Fourier series a bit more formally, including how to compute them.

Note. A *Fourier series* is a series of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{\ell}\right) + b_m \sin\left(\frac{m\pi x}{\ell}\right)\right).$$

Definition. Let u and v be real-valued functions defined for $\alpha \le x \le \beta$. Define the *inner product* of u and v on the interval $[\alpha, \beta]$ to be

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx.$$

Functions u and v are said to be *orthogonal* on the interval $x \in [\alpha, \beta]$ if

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx = 0.$$

We can verify that $\cos(m\pi x/\ell)$ and $\sin(m\pi x/\ell)$ are orthogonal for $x \in [-\ell, \ell]$ for all m and n. Also

$$\left(\cos\left(\frac{m\pi x}{\ell}\right), \cos\left(\frac{n\pi x}{\ell}\right)\right) = \left(\sin\left(\frac{m\pi x}{\ell}\right), \sin\left(\frac{n\pi x}{\ell}\right)\right) = \begin{cases} 0 \text{ if } m \ne n \\ \ell \text{ if } m = n. \end{cases}$$

Note. A computation based on this orthogonality principle reveals (see page 523) that is

$$f(x) = \frac{1_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{\ell}\right) + b_m \sin\left(\frac{m\pi x}{\ell}\right)\right)$$

then

$$a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{m\pi x}{\ell}\right)\,dx$$
$$b_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{m\pi x}{\ell}\right)\,dx.$$

Theorem 10.3.1. (Fourier) Suppose that f and f' are piecewise continuous for $x \in [-\ell, \ell]$. Also suppose f is defined outside this interval and is periodic with period 2ℓ. Then $f(x)$ has the Fourier series above and the coefficients a_m and b_m (called *Fourier coefficients*) are as above. The series converges at all points where f is continuous and at the points of discontinuity (say x_0) it converges to

$$\frac{\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)}{2}.$$

Note. To illustrate the previous theorem, here is a graph of the 8th partial sum of a Fourier series for a step function. This is Figure 10.3.3 in the 10 edition of DiPrima and Boyce: