Chapter 6. The Laplace Transform

Section 6.1. Definition of the Laplace Transform

Note. In this section we define a transform that will be useful in solving DEs. We will take the DE, transform it, solve the transformed DE, and then transform back to the solution of the original DE.

Definition. A *integral transform* of $f(x)$ is a relation of th eform

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t)\, dt.$$

$K(s,t)$ is the *kernel* of the transformation.

Definition. The *Laplace transform* of $f(x)$ is

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt.$$

Definition. Suppose there exists constants K, a, and M such that $|f(t)| \leq Ke^{1t}$ for all $t > M$. Then $f(t)$ is of *exponential order*.

Theorem 6.1.2. If f is (piecewise) continuous on the interval $0 \leq r \leq A$ for any positive A and if $f(x)$ is of exponential order, then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$

Example. Page 248 Example 4. Find the Laplace transform of $f(x) = 1$.

Solution. We have

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{R\to\infty}\int_0^R e^{-st}\,dt = \lim_{r\to\infty} -\frac{1}{2}e^{-st}\Big|_{t=0}^{t=R}$$

$$= \lim_{r\to\infty}\left(-\frac{1}{s}e^{-sR} + \frac{1}{s}\right) = \frac{1}{s}.$$

Example. Find the Laplace transform of $f(x) = x$.

Solution. We have

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,dt = \lim_{R\to\infty}\int_0^R te^{-st}\,dt = \lim_{R\to\infty}\left(-\frac{t}{s}e^{-st}\Big|_{t=0}^{t=R} - \int_0^R -\frac{1}{s}e^{-st}\,dt\right)$$

$$= \lim_{R\to\infty}\left(-\frac{t}{s}e^{-st}\Big|_{t=0}^{t=R} - \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=R}\right) = (0-0) - \left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2}.$$

Note. Notice that if the Laplace transform of f_1 and f_2 exists, then

$$\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = \int_0^\infty e^{st}(c_1f_1(t) + c_2f_2(t))\,dt$$

$$= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.$$

So $\mathcal{L}$ is a *linear operator* (as is also, for example, differentiation).

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