Section 6.2. Solutions of Initial Value Problems

Note. In this section we will use the Laplace transform to solve IVPs for linear DEs with constant coefficients.

Theorem 6.2.1. Suppose f and f' are continuous on any interval $0 \le t \le A$ and that $f(t)$ is of exponential order. Then $\mathcal{L}\{f(t)\}$ exists for $s > a$ (see the definition of exponential order) and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof. We have

$$\begin{aligned} \int_0^A e^{-st} f'(t)\, dt &= \left(e^{-st} f(t) + s \int e^{-st} f(t)\, dt \right) \Bigg|_{t=0}^{t=A} \\ &\quad \text{by integration by parts with } u = e^{-st} \text{ and } dv = f'(t)\, dt \\ &= e^{-sA} f(A) - f(0) + s \int_0^A d^{-st} f(t)\, dt. \end{aligned}$$

Letting $A \to \infty$ gives $\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}$ since $e^{-sA} f(A) \to 0$ when $s > a$. ∎

Note. A corollary to Theorem 6.2.1 which can be proved by Mathematical Induction is the following.

Corollary. Suppose $f, f', \ldots, f^{(n)}$ are continuous on any interval $0 \le t \le A$. Suppose that $f, f', \ldots, f^{(n-1)}$ are of exponential order. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) = \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Example. Page 289 Number 12. Use Laplace transforms to solve the IVP:

$$\begin{cases} y'' + 3y' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Solution. By the linearity of the Laplace transform, we have

$$\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{0\}.$$

This implies, by Corollary,

$$\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + 3\left(s\mathcal{L}\{y\} - y(0)\right) + 2\mathcal{L}\{y\} = 0.$$

Let $Y = \mathcal{L}\{y\}$ and then we have

$$s^2Y - s - 3sY - 3 + 2Y = 0 \text{ or } Y(s^2 + 3s + 2) = s + 3$$

or

$$Y = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+2)(s+1)} = \frac{-1}{s+2} + \frac{2}{s+1}.$$

So the solution is $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{-\frac{1}{s+2} + \frac{2}{s+1}\right\}$. Notice that

$$\mathcal{L}\{e^{-2t}\} = \int_0^\infty e^{-st}e^{-2t}\,dt = \int_0^\infty e^{-t(s+2)}\,dt = \frac{1}{s+2}$$

and similarly $\mathcal{L}\{e^{-t}\} = 1/(s+1)$. So

$$\mathcal{L}^{-1}\left\{-\frac{1}{s+2} + \frac{2}{s+1}\right\} = -e^{2t} + 2e^{t}.$$

Note. Notice that we convert the DE into an algebraic equation which we solve. We (unfortunately) then have to calculate an inverse Laplace transform.

Example. Page 290 Number 28. Suppose $F(s) = \int_0^\infty e^{-st} f(t)\, dt$.
(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.

Solution. We have

$$\begin{aligned} \frac{d}{ds}[F(s)] &= \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right] = \int_0^\infty \frac{d}{ds}\left[e^{-st} f(t)\right]\, dt \\ &= \int_0^\infty -te^{-st} f(t)\, dt = \int_0^\infty e^{-st}(-tf(t))\, dt = \mathcal{L}\{-tf(t)\}. \end{aligned}$$

(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)f(t)\}$.

Solution. We will demonstrate this by mathematical induction. The result is true for $n = 1$ by part (a). Now suppose it is true for $n = k$; that is, $F^{(k)}(s) = \mathcal{L}\{(-t)^k f(t)\}$. We must show that this implies the result for $n = k + 1$:

$$\begin{aligned} F^{(k+1)}(s) &= \frac{d}{ds}[F^{(k)}(s)] = \frac{d}{ds}[\mathcal{L}\{(-t)^k f(t)\} = \frac{d}{ds}\left[\int_0^\infty e^{-st}(-t)^k f(t)\, dt\right] \\ &= \int_0^\infty \frac{d}{ds}[e^{-st}(-t)^k f(t)\, dt] = \int_0^\infty e^{-st}(-t)^{k+1} f(t)\, dt = \mathcal{L}\{(-t)^{k+1} f(t)\}. \end{aligned}$$

Therefore, by Mathematical Induction, the result follows.

Example. Page 290 Number 30. Evaluate $\mathcal{L}\{t^2 \sin bt\}$.

Solution. With $n = 2$ in Number 28(b),

$$\mathcal{L}\{t^2 \sin bt\} = \frac{d^2}{ds^2}[\mathcal{L}\{\sin bt\} = \frac{d^2}{ds^2}\left[\frac{b}{s^2+b^2}\right] = \frac{d}{ds}\left[\frac{-2sb}{(s^2+b^2)^2}\right] = \frac{6sb^2 - 2b^3}{(s^2+b^2)^3}.$$

Notice that we have exchanged calculating Laplace transforms of "known" functions times powers of t for differentiation of the Laplace transform of the "known" functions.

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