Chapter 7. Systems of First Order Linear Equations

Note. In this chapter we study a system of several simultaneous linear ODEs involving several dependent variables, each a function of a single dependent variable. Such systems arise in the study of electrical circuits, mechanical systems, and ecological models.

Section 7.1. Introduction

Note. In this section we motivate the study of systems of differential equations with physical models based on masses and springs. We also give some existence and uniqueness results concerning systems of ODEs.

Example. Suppose we have the following mechanical system:

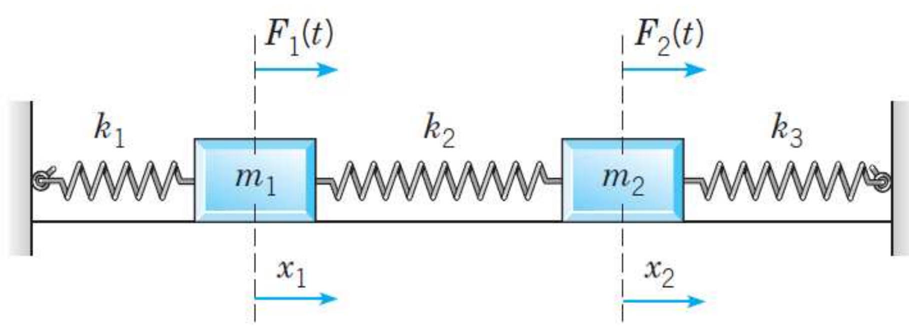

Figure 7.1.1 from the 10th edition of DiPrima and Boyce

Suppose the system is set up such that when it is at rest, each spring is at its natural length. Recall that the force exerted by a spring with spring constant k

which is displaced a distance x from its natural length is $F = kx$ by Hooke's Law. Suppose this system is perturbed. let x_1 be the distance spring 1 is *stretched* and let x_2 be the distance spring 3 is *compressed*. Find the system of DEs for x_1 and x_2. Notice that there are no external forces on the masses. This is not the case on page 381.

Solution. By Newton's Law, force = mass $\times$ acceleration. So we have

$$\begin{aligned} m_1 \frac{d^2 x_1}{xt^2} &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 \frac{d^2 x_2}{dt^2} &= -k_2(x_2 - x_1) - k_3 x_2. \end{aligned}$$

So we have the system of equations:

$$\begin{cases} m_1 x_1'' + (k_1 + k_2)x_1 - k_2 x_2 = 0 \\ m_2 x_2'' + (k_2 + k_3)x_2 - k_2 x_1 = 0. \end{cases}$$

Notice this is a system of two second order linear DEs in the dependent variables x_1 and x_2 and the independent variable is t.

Example. We now consider a mass on a vertical spring. Suppose the mass m stretches the spring a distance L under the influence of gravity. By Hooke's Law, if the spring constant is k and the acceleration due to gravity is g, then $mg = kL$. Let $u(t)$ be the position of the mass at time t where $u(t)$ is measured from the equilibrium position. Newton's Law of Motion says that force = mass $\times$ acceleration. So, $mu''(t)$ = the force on m. There are four forces on m:

1. weight $= mg$,

2. spring force $= -k(L + u(t))$,

3. the damping force due to resistance of the medium; this force is proportional to the velocity and equals $-\gamma u'(t)$, and

4. any applied external force $F(t)$.

So the DE describing the motion is

$$mu''(t) = mg - k(L + u(t)) - \gamma u'(t) + F(t)$$

or

$$mu''(t) + \gamma u'(t) + ku(t) = F(t).$$

Note. Notice that with the proper substitution, the 2nd order linear DE above can be converted to a system of 2 first order linear DEs. Let $x_1 = u$ and $x_2 = u'$. Then we get

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 + \frac{F(t)}{m}. \end{cases}$$

In fact, any nth order DE of the form

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$$

can be reduced to a system of n first order equations. Let

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ &\vdots \\ x_n &= y^{(n-1)}. \end{aligned}$$

Then we get

$$\begin{aligned}
x_1' &= x_2 \\
x_2' &= x_3 \\
x_3' &= x_4 \\
&\vdots \\
x_{n-1}' &= x_n \\
x_n' &= F(t, x_1, x_2, \ldots, x_n).
\end{aligned}$$

Note. We now state a theorem that, crudely put, guarantees the existence of a unique solution for a system of differential equations.

Theorem 7.1.1. Basic Theorem on Existence and Uniqueness for a System of DEs.

Consider the system of DEs

$$\begin{cases}
x_1' = F_1(t, x_1, x_2, \ldots, x_n) \\
x_2' = F_2(t, x_1, x_2, \ldots, x_n) \\
\qquad\qquad \vdots \\
x_n' = F_n(t, x_1, x_2, \ldots, x_n)
\end{cases}$$

with the initial conditions

$$\begin{cases}
x_1(t_0) = x_1^0 \\
x_2(t_0) = x_2^0 \\
\qquad \vdots \\
x_n(t_0) = x_n^0.
\end{cases}$$

Let each of the function s$F_1, F_2, \ldots, F_n$ and the partial derivatives

$$\frac{\partial F_1}{\partial x_1}, \frac{\partial F_1}{\partial x_2}, \ldots, \frac{\partial F_1}{\partial x_n}$$
$$\frac{\partial F_2}{\partial x_1}, \frac{\partial F_2}{\partial x_2}, \ldots, \frac{\partial F_2}{\partial x_n}$$
$$\vdots$$
$$\frac{\partial F_n}{\partial x_1}, \frac{\partial F_n}{\partial x_2}, \ldots, \frac{\partial F_n}{\partial x_n}$$

be continuous in a region R of $n+1$ dimensional space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, $\alpha_2 < x_2 < \beta_2$, $\ldots$, $\alpha_n < x_n < \beta_n$ and let the point $(t_0, x_1^0, x_2^0, \ldots, x_n^0) \in R$. Then there is an interval $|t - t_0| < h$ in which there exists a unique solution of the system of DEs that satisfies the initial conditions (see Theorem 2.4.1 for a special case of this).

Definition. If each of the functions above, $F_1, F_2, \ldots, F_n$ is a linear function of each of the variables $x_1, x_2, \ldots, x_n$ then the systems of equations is said to be *linear*. Otherwise the system is *nonlinear*.

Note. A function $F(x_1, x_2, \ldots, x_n)$ is continuous in x_1 (for example) if

$$F(a+b, x_2, \ldots, x_n) = F(a, x_2, \ldots, x_n) + F(b, x_2, \ldots, x_n)$$

and $F(cx_1, x_2, \ldots, x_n) = cF(x_1, x_2, \ldots, x_n)$.

Definition/Note. The most general system of n first order linear DEs is

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\ \vdots \\ \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t). \end{cases}$$

If each of th e$g_i(t)$ is identically zero then the system is *homogeneous*, otherwise it is *nonhomogeneous*.

Note. We now state an existence and uniqueness theorem for this type of system of DEs.

Theorem 7.1.2. Basic Existence and Uniqueness Theorem for a Linear System of DEs.

Of each of the p_{ij} and g_i are continuous on an open interval $I : \alpha < t < \beta$ in the system of first order linear DEs above, then there exists a unique solution of the system that satisfies the initial conditions

$$\begin{cases} x_1(t_0) = x_1^0 \\ x_2(t_0) = x_2^0 \\ \vdots \\ x_n(t_0) = x_n^0. \end{cases}$$

where $t_0 \in I$. In fact, the solution exists throughout the interval I.

Note. Notice that a system of linear DEs could be nicely described using matrices. We shall take advantage of this soon.

Revised: 3/4/2019