# **Section 7.2.** Review of Matrices

**Note.** We quickly review some basic ideas concerning matrices and vectors.

**Note.** The book represents matrices with bold faced letters. I will represent them with capital letters, as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Matrix $A$ is $m \times n$ with $m$ *rows* and $n$ *columns*. We express $A$ briefly as $A = [a_{ij}]$. We will not assume that the entries of the matrices are real.

**Definition.** For a complex number $z = a + bi \in \mathbb{C}$ (where $a$ and $b$ are real) define the *conjugate* of $z$ as $\overline{z} = a - bi$.

**Definition.** The *transpose* of matrix $A$, denoted $A^T$, is the matrix obtained from $A$ by interchanging the rows and columns of $A$. Notice that if $A$ is $m \times n$ then $A^T$ is $n \times m$. If $A = [a_{ij}]$ then $A^T = [a_{ji}]$. If $A = [a_{ij}]$ then the *conjugate* of $A$, denoted $\overline{A}$, is $\overline{A} = [\overline{a_{ij}}]$. The matrix $\overline{A}^T$ is the *adjoint* of $A$ and is denoted $A^*$.

**Note.** Recall that the addition of matrices (done component-wise) behaves as you would expect (commutativity, associativity, and scalars multiplication distributes over matrix addition).

**Definition.** If $A$ is $m \times n$ and $B$ is $n \times R$, then define the *product* of $A$ and $B$, $C = AB$, to be the matrix $C$ such that $c_{ij}$ is found by multiplying the $i$th row of $A$ by the $j$th row of $B$ and adding the resulting products:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

**Note.** Recall matrix multiplication is *associative*, $A(BC) = (AB)C$. Matrix multiplication distributes over matrix addition, $A(B + C) = AB + AC$. However, it is not in general true that matrix multiplication is *commutative*. In fact, a certain (restricted) collection of matrices form a *nonabelian* (i.e., noncommutative) *group* under matrix multiplication.

**Note.** The "multiplication" of vectors can be considered a special case of matrix multiplication.

**Definition.** For two vectors $\vec{x}$ and $\vec{y}$ (both $n \times 1$), define the *inner product* (or *scalar product*) to be

$$(\vec{x}, \vec{y}) = \sum_{i=1}^{n} x_i \overline{y_i} = (\vec{x})^T \overline{\vec{y}}.$$

**Note.** We can verify the following:

$$(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})}$$

$$(\alpha\vec{x}, \vec{y}) = \alpha(\vec{x}, \vec{y})$$

$$(\vec{x}, \vec{y} + \vec{z}) = (\vec{x}, \vec{y} + (\vec{x}, \vec{z})$$

$$(\vec{x}, \alpha\vec{y}) = \overline{\alpha}(\vec{x}, \vec{y}).$$

**Definition.** The *length* or *magnitude* of a vector $\vec{x}$ is

$$\|\vec{x}\| = (\vec{x}, \vec{x})^{1/2} = \left(\sum_{i=1}^{n} x_i \overline{x_i}\right)^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

If $(\vec{x}, \vec{y}) = 0$, then $\vec{x}$ and $\vec{y}$ are *orthogonal*.

**Definition.** The $n \times n$ *identity matrix* is

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

**Note.** For all $n \times n$ matrices, $\mathcal{I}A = A\mathcal{I} = A$.

**Definition.** If $A$ is a $n \times n$ matrix and there is a matrix $B$ such that $AB = BA = \mathcal{I}$, then $B$ is the *inverse* of $A$, denoted $B = A^{-1}$. If there is such an inverse for $A$ then $A$ is *nonsingular*. If $A$ has no inverse, it is called *singular*.

**Theorem.** $A$ is nonsingular if and only if the determinant of $A$ is not 0.

**Definition.** An *elementary row operation* is one of the following:

**1.** interchanging two rows,

**2.** multiplication of a row by a nonzero scalar,

**3.** addition of a multiple of one row to another row.

**Note.** Recall that if $\det(A) \neq 0$ then $A$ can be transformed into $\mathcal{I}$ with a sequence of elementary row operations. If the same sequence of row operations is applied to $\mathcal{I}$, the result is $A^{-1}$.

**Example.** If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ then $A^{-1} = ?$

**Note.** We can also define matrix functions, or actually a function of matrices. As you expect, integration and differentiation of matrix functions are done component-wise. In fact, the Product Rule still holds:

$$\frac{d}{dt}[AB] = A\frac{dB}{dt} + \frac{dA}{dt}B$$

where $A$ and $B$ are matrix functions of $t$. So we can also have differential equations of matrix functions!

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