Section 7.3. Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors

Note. In this section, we review a bunch of material from sophomore Linear Algebra (MATH 2010). In addition to the topics mentioned in the title of this section, we cover adjoints, similar matrices, and diagonal matrices.

Note. Notice that the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

can be written as a matrix equation $A\vec{x} = \vec{b}$. If A is nonsingular, then the system has a unique solution given by $\vec{x} = A^{-1}\vec{b}$. If A is singular, then the system either has no solution (and is "inconsistent") or has an infinite number of solutions.

Note. As opposed to finding A^{-1}, there is a computational shortcut. We can create the *augmented matrix* $[A \mid \vec{b}]$.

Example. If

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ & & 2x_2 & - & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & & & = & 8 \end{array}$$

then find x_1, x_2, x_3. Use A^{-1} from the example of the previous section. (This has a unique solution.)

Example. If

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 3 \\ 2x_1 & - & x_2 & - & 3x_3 & = & -2 \\ 3x_1 & & & - & 2x_3 & = & 0 \end{array}$$

then find x_1, x_2, x_3. (This has no solution.)

Example. If

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 3 \\ 2x_1 & - & x_2 & - & 3x_3 & = & -2 \\ 3x_1 & & & - & 2x_3 & = & 1 \end{array}$$

then find x_1, x_2, x_3. (This has multiple solutions.)

Note. Recall the computation of the determinants for 2×2 and 3×3 matrices:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Definition. As with a set of functions, a set of vectors $\{\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(k)}\}$ is *linearly dependent* if

$$c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_k\vec{x}^{(k)} = \vec{0}$$

for some collection of scalars, at least one of which is nonzero. Otherwise the vectors are *linearly independent*.

Note. If we use the above vectors $\vec{x}^{(i)}$ to build a matrix X with ith column $\vec{x}^{(i)}$, we can write the vector equation as a matrix equation $X\vec{c} = \vec{0}$. Certainly $\vec{c} = \vec{0}$ is a solution to this matrix equation. In fact, if $\det(X) \neq 0$ then the solution is unique. Therefore, the $\vec{x}^{(i)}$s are linearly independent if and only if $\det(X) \neq 0$ where X. We can also potentially use this method to check for the linear independence of functions.

Example. Page 343 Number 7. Determine whether the vectors $\vec{x}^{(1)} = [2,1,0]$, $\vec{x}^{(2)} = [0,1,0]$, $\vec{x}^{(3)} = [-1,2,0]$ are linearly independent. If they are linearly dependent, find the linear relation among them.

Solution. We consider the equation $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + c_3\vec{x}^{(3)} = \vec{0}$. This gives:

$$c_1[2,1,0] + c_2[0,1,0] + c_3[-1,2,0] = [2c_1 - c_3, c_2 + c_2 + 2c_3, 0] = [0,0,0]$$

so we consider the system of equations

$$\begin{array}{rcrcrcl} 2c_1 & & & - & c_3 & = & 0 \\ c_1 & + & c_2 & + & 2c_3 & = & 0 \\ & & & & 0 & = & 0. \end{array}$$

This leads to the augmented matrix which we row reduce:

$$\left[\begin{array}{rrr|r} 2 & 0 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + (1/2)R_2} \left[\begin{array}{rrr|r} 1 & 0 & -1/2 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to -(1/2)R_2} \left[\begin{array}{rrr|r} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so we need $\begin{array}{rcrcrcl} c_1 & & & - & (1/2)c_3 & = & 0 \\ & & c_2 & + & (5/2)c_3 & = & 0 \\ & & & & 0 & = & 0 \end{array}$ or $\begin{array}{rcr} c_1 & = & (1/2)c_3 \\ c_2 & = & -(5/2)c_3 \\ c_3 & = & c_3 \end{array}$ or with $t = c_3$ as a free variable, $\begin{array}{rcr} c_1 & = & (1/2)t \\ c_2 & = & -(5/2)t \\ c_3 & = & t. \end{array}$ So the vectors are linearly dependent. We can take (with $t = 2$) $c_1 = 1$, $c_2 = -5$, and $c_3 = 2$ to get

$$\vec{x}^{(1)} - 5\vec{x}^{(2)} + 2\vec{x}^{(3)} = [2, 1, 0] - 5[0, 1, 0] + 2[-1, 2, 0] = [0, 0, 0].$$

Note. An $n \times n$ matrix can be viewed as a linear transformation (rotation, translation, and "stretching") of an n-dimensional vector, $A\vec{x} = \vec{y}$, $\vec{x}$ is transformed to $\vec{y}$ by A. Vectors that are transformed to multiples of themselves are of particular interest, i.e. $A\vec{x} - \lambda\vec{x} = \vec{0}$ or $(A - \lambda\mathcal{I})\vec{x} = \vec{0}$. As above, this equation has nonzero solutions only if $\det(A - \lambda\mathcal{I}) = 0$.

Definition. For $n \times n$ matrix A and $n \times n$ identity matrix $\mathcal{I}$, values of λ satisfying the equation $\det(A - \lambda\mathcal{I}) = 0$ are called *eigenvalues* of A. For an eigenvalue λ of A, a nonzero vector $\vec{x}$ satisfying $A\vec{x} = \lambda\vec{x}$ is called an *eigenvector* of λ.

Definition/Note. Eigenvectors of an eigenvalue are only determined up to a constant multiple. If an eigenvector is chosen such that its length is one, it is said to be *normalized*.

Example. Page 344 Number 21. Find the eigenvalues and all eigenvectors of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}.$$

Solution. We consider

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)^2 + 4) = (1-\lambda)(5 - 2\lambda + \lambda^2).$$

Setting $\det(A - \lambda\mathcal{I}) = 0$ we have the eigenvalues $\lambda_1 = 1$, $\lambda_2 = (2 + \sqrt{4-20})2 = 1 + 2i$, and $\lambda_3 = (2 - \sqrt{4-20})/2 = 1 - 2i$. We now find the corresponding eigenvectors.

$\underline{\lambda_1 = 1.}$ We solve for the components of $\vec{x} = [x_1, x_2, x_3]$ such that $(A - 1\mathcal{I})\vec{x} = \vec{0}$, which gives the augmented matrix:

$$\left[\begin{array}{ccc|c} 1-(1) & 0 & 0 & 0 \\ 2 & 1-(1) & -2 & 0 \\ 3 & 2 & 1-(1) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right] \underset{\sim}{\scriptstyle R_3 \to R_3 - R_2}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \underset{\sim}{\scriptstyle R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{\scriptstyle R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -4 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\sim}{\scriptstyle R_2 \to R_2/(-4)} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{\scriptstyle R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so we need $\begin{array}{rcrcl} x_1 & & - & x_3 & = & 0 \\ & x_2 & + & (3/2)x_3 & = & 0 \\ & & & 0 & = & 0 \end{array}$ or $\begin{array}{rcl} x_1 & = & x_3 \\ x_2 & = & -(3/2)x_3 \\ x_3 & = & x_3 \end{array}$ or with $t = x_3/2$ as a free variable, $\begin{array}{rcl} x_1 & = & 2t \\ x_2 & = & -3t \\ x_3 & = & 2t. \end{array}$ So the set of eigenvectors associated with $\lambda_1 = 1$ are

$$\left\{ t \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \;\middle|\; t \in \mathbb{R}, t \neq 0 \right\}.$$

$\underline{\lambda_2 = 1+2i.}$ We solve for the components of $\vec{x} = [x_1, x_2, x_3]$ such that $(A - (1+2i)\mathcal{I})\vec{x} = \vec{0}$, which gives the augmented matrix:

$$\left[\begin{array}{ccc|c} 1-(1+2i) & 0 & 0 & 0 \\ 2 & 1-(1+2i) & -2 & 0 \\ 3 & 2 & 1-(1+2i) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & 2i & 0 \end{array}\right]$$

$$\overset{R_1 \to R_1/(-2i)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & 2i & 0 \end{array}\right] \underset{R_3 \to R_3 - 3R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 2 & 2i & 0 \end{array}\right]$$

$$\overset{R_2 \to R_2/(-2i)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 2 & 2i & 0 \end{array}\right] \overset{R_3 \to R_3 - 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so we need $\begin{array}{rcl} x_1 & = & 0 \\ x_2 - ix_3 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & ix_3 \\ x_3 & = & x_3 \end{array}$ or with $t = x_3$ as a free variable,

$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & it \\ x_3 & = & t. \end{array}$ So the set of eigenvectors associated with $\lambda_1 = 1+2i$ are

$$\left\{ t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R}, t \neq 0 \right\}.$$

$\underline{\lambda_3 = 1 - 2i.}$ We solve for the components of $\vec{x} = [x_1, x_2, x_3]$ such that $(A - (1 - 2i)\mathcal{I})\vec{x} = \vec{0}$. The technique is similar to above and we find

$$\left\{ t \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \middle| \; t \in \mathbb{R}, t \neq 0 \right\}.$$

Definition. A matrix A is *self adjoint* (or *Hermitian*) if it satisfies $A^* = A$. If A contains only real entries, this simply means that A is *symmetric*, $A^T = A$.

Theorem. If A is Hermitian, then

1. all the eigenvalues of A are real,

2. A has a set of associated eigenvectors of size n (where A is $n \times n$) which are linearly independent,

3. eigenvectors of different eigenvalues of A are orthogonal, and

4. the set of n eigenvectors can be chosen to all be mutually orthogonal as well as linearly independent.

Example. Page 344 Numbers 27 and 32. Prove that if A is Hermitian, then $(A\vec{x}, \vec{y}) = (\vec{x}, A\vec{y})$.

Solution. From the definition of matrix product, for $n \times n$ matrix $A = [a_{ij}]$ and with the components of column vector $\vec{x}$ as x_i, we have

$$A\vec{x} = \begin{bmatrix} \sum_{i=1}^n a_{1i}x_i \\ \sum_{i=1}^n a_{2i}x_i \\ \vdots \\ \sum_{i=1}^n a_{ni}x_i \end{bmatrix}$$

and $(A\vec{x}, \vec{y} = (A\vec{x})^T\overline{(\vec{y})}$ so

$$\begin{aligned} (A\vec{x}, \vec{y}) &= \overline{y}_1 \left(\sum_{i=1}^n a_{1i}x_i \right) + \overline{y}_2 \left(\sum_{i=1}^n a_{2i}x_i \right) + \cdots + \overline{y}_n \left(\sum_{i=1}^n a_{ni}x_i \right) \\ &= \sum_{j=1}^n \overline{y}_j \left(\sum_{i=1}^n a_{ji}x_i \right) = \sum_{i=1}^n x_i \left(\sum_{j=1}^n a_{ji}\overline{y}_j \right) \\ &= x_1 \left(\sum_{j=1}^n a_{j1}\overline{y}_1 \right) + x_2 \left(\sum_{j=1}^n a_{j2}\overline{y}_2 \right) + \cdots + x_n \left(\sum_{j=1}^n a_{jn}\overline{y}_n \right) \\ &= \vec{x}^T(A^T\overline{(\vec{y})}) = \vec{x}^T\overline{(\overline{A^T}\vec{y})} = \vec{x}^T\overline{(A^*\vec{y})} = (\vec{x}, A^*\vec{y}). \end{aligned}$$

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Note. Since matrix multiplication is defined in a rather complicated way, computing powers of matrices is rather difficult (except, I guess, for $\mathcal{I}$).

Definition. A matrix D is a *diagonal matrix* if all the entries off the diagonal are 0.

Note. It is easy to raise a diagonal matrix to various powers:

$$D^m = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}^m = \begin{bmatrix} d_1^m & 0 & \cdots & 0 \\ 0 & d_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^m \end{bmatrix}.$$

In fact, we *will* need to raise matrices to powers and use them in power series.

Definition. Matrix A is *similar* to matrix B if there exists an invertible matrix T such that $B = T^{-1}AT$. If A is similar to a diagonal matrix, then A is *diagonalizable*.

Note. It is fairly easy to raise a diagonalizable matrix to a power:

$$A^m = (T^{-1}DT)^m = (T^{-1}DT)(T^{-1}DT)\cdots(T^{-1}DT) = T^{-1}D^mT.$$

Theorem. If A has a full set of n linearly independent eigenvectors (notice that this is the case if A is Hermitian), say $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}$, and T is the matrix whose columns are these eigenvectors, then $D = T^{-1}AT$ where $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

Note. Since the columns of T in the above theorem are linear independent, $\det(T) \neq 0$ and T^{-1} exists.

Theorem. If A has fewer than n linearly independent eigenvectors, then there is no such T as described above, that is, A is not diagonalizable.

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