Section 7.4. Basic Theory of Systems of First Order Linear Equations

Note. We convert a system of first order linear DEs into matrix form and give some theoretical results concerning solutions.

Note. A system of n first order linear DEs:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

can be written in matrix form as

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t).$$

We have seen that we can convert an nth order linear DE with constant coefficients into a system of n first order linear DEs with constant coefficients. We solved this former type of DE in introduction to Differential Equations; see Chapters 3 and 4 of my online notes at

`http://faculty.etsu.edu/gardnerr/Differential-Equations/DE-Ross4-notes.htm`.

The methods we use to solve the latter type of system of DEs will be similar to the method seen in introductory Differential Equations (MATH 2120).

Theorem 7.4.1. Principle of Superposition (for Systems).
If $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are solutions of

$$\vec{x}' = P(t)\vec{x}$$

then any linear combination $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is also a solution.

Theorem 7.4.2. If $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}$ are linearly independent solutions of

$$\vec{x}' = P(t)\vec{x}$$

for each point in $\alpha < t < \beta$, then each solution $\vec{x} = \vec{\varphi}(t)$ of the DE can be expressed as a linear combination of the $\vec{x}^{(i)}$'s:

$$\vec{\varphi}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) + \cdots + c_n\vec{x}^{(n)}(t)$$

in exactly one way.

Definition. The *general solution* of $\vec{x}' = P(t)\vec{x}$ is

$$c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) + \cdots + c_n\vec{x}^{(n)}(t)$$

where c_i are arbitrary and $\vec{x}^{(i)}$ are as described in Theorem 7.4.2. Any set of n linearly independent solutions to the DE (on an interval $\alpha < t < \beta$) is a *fundamental set of solutions* (for that interval).

Definition. For a set of n solutions $\vec{x}^{(i)}$, $i = 1, 2, \ldots, n$ of the system $\vec{x}' = P(t)\vec{x}$, from the matrix $\vec{X}(t)$ by making column i of $\vec{X}(t)$ as $\vec{x}^{(i)}$. Then the *Wronskian* of these n solutions is

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}] = \det(\vec{X}).$$

Note. The columns of a matrix are linearly independent if and only if the determinant of the matrix is nonzero (for all values of t in a given interval). So $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}$ is a fundamental set of solutions to $\vec{x}' = P(t)\vec{x}$ in the interval $\alpha < t < \beta$ if and only if the Wronskian is nonzero for $\alpha < t < \beta$.

Theorem 7.4.3. If $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}$ are solutions of $\vec{x}' = P(t)\vec{x}$ in the interval $\alpha < t < \beta$ then in this interval, the Wronskian $W[\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}]$ is either identically zero or never zero in this interval.

Note. From Theorem 7.4.3, to test $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}$ as a fundamental set of solutions to $\vec{x}' = P(t)\vec{x}$ in the interval $\alpha < t < \beta$, we need only check the Wronskian at one point, say t_0, of the interval (α, β).

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