Section 7.5. Homogeneous Linear Systems with Constant Coefficients

Note. We now use matrix techniques to solve particular IVPs.

Note. Recall that the solution to the first order homogeneous linear IVP

$$\begin{cases} x' = ax \\ x(0) = x_0 \end{cases}$$

is $x = x_0 e^{at}$. So, in solving the linear homogeneous system of first order DEs with constant coefficients

$$\vec{x}' = A\vec{x}$$

we seek a solution of the form $\vec{x} = \vec{\xi} e^{Rt}$ where $\vec{\xi}$ is a constant vector. We have $\vec{x}' = \vec{\xi} E e^{Rt}$ and the DE becomes

$$\vec{\xi} R e^{RT} = A\vec{\xi} e^{Rt} \text{ or } R\vec{\xi} = A\vec{\xi} \text{ or } A\vec{\xi} - R\vec{\xi} = \vec{0} \text{ or } (A - R\mathcal{I})\vec{\xi} = \vec{0}.$$

So we have such a solution if R is an eigenvalue of A and $\vec{\xi}$ is an eigenvector.

Example. Page 356 Number 2. Consider

$$\vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x}.$$

Find the general solution and draw trajectories in the x_1, x_2-plane of solutions.

Solution. From the note above, we need $(A - R\mathcal{I})\vec{\xi} = \vec{0}$. So consider

$$\begin{vmatrix} 1-R & -2 \\ 3 & -4-R \end{vmatrix} = -(1-R)(4+R)+6 = R^2+3R+2 = (R+2)(R+1),$$

so we must have $R = -2$ or $R = -1$. With $R = -2$, equation $(A - (-2)\mathcal{I})\vec{\xi} = \vec{0}$ has augmented matrix

$$\left[\begin{array}{cc|c} 1-(-2) & -2 & 0 \\ 3 & -4-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right] \underset{\sim}{\overset{R_2 \to R_2 - R_1}{}} \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and we choose the eigenvector and solution corresponding to $R = -2$ of

$$\vec{\xi}^{(1)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \vec{x}^{(1)} = e^{-2t}\vec{\xi}^{(1)} = \begin{bmatrix} 2e^{-2t} \\ 3e^{-2t} \end{bmatrix}.$$

With $R = -1$, equation $(A - (-1)\mathcal{I})\vec{\xi} = \vec{0}$ has augmented matrix

$$\left[\begin{array}{cc|c} 1-(-1) & -2 & 0 \\ 3 & -4-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right] \underset{\sim}{\overset{R_2 \to R_2 - (3/2)R_1}{}} \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and we choose the eigenvector and solution corresponding to $R = -1$ of

$$\vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{x}^{(2)} = e^{-t}\vec{\xi}^{(2)} = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}.$$

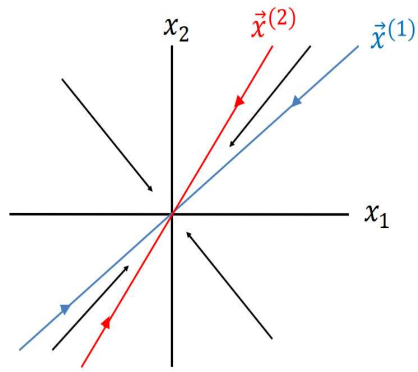
Notice that the Wronskian is

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \begin{vmatrix} 2e^{-2t} & e^{-t} \\ 3e^{-2t} & e^{-t} \end{vmatrix} = (2e^{-2t})(e^{-t}) - (e^{-t})(3e^{-2t}) = -e^{-3t} \neq 0,$$

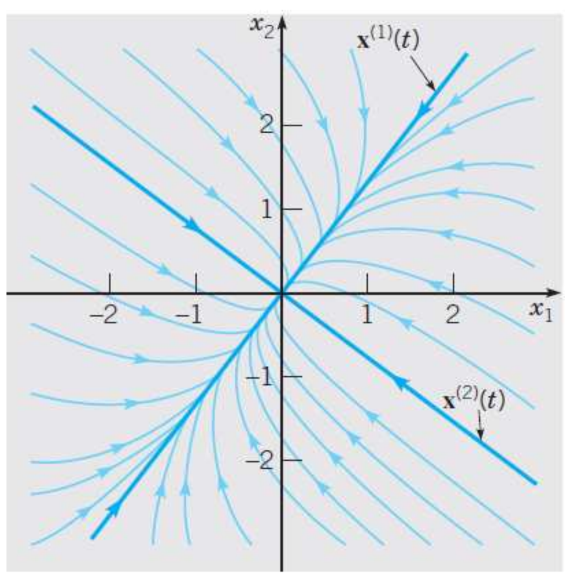
so $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are linearly independent. So the general solution is

$$\vec{x} = c_1 \begin{bmatrix} 2e^{-2t} \\ 3e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}.$$

If we let $\vec{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $x_1 = 2e^{-2t}$ and $x_2 = 3e^{-2t}$. With $\vec{x}^{(2)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 = x_2 = e^{-t}$. So in the x_1, x_2-plane we have (as t increases):



A linear combination of $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$, say $\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ can be represented as a point in the x_1, x_2-plane. We can use $\vec{x}^{(1)}$ an d$\vec{x}^{(2)}$ to illustrate the behavior of a solution $\vec{x}$ as time increases. Notice that all trajectories approach 0, but trajectories along $\vec{x}^{(1)}$ approach 0 faster (due to the e^{-2t} term) than trajectories along $\vec{x}^{(2)}$ (which involve the term e^{-t}). This is not reflected in the diagram above. Consider the following more detailed diagram for a different problem.



This diagram (Figure 7.5.4(a) from the 10th edition of DePrima and Boyce) reflects trajectories for different values of c_1 and c_2. These trajectories are for

$$\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} = c_1 e^{-t}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

Notice that in this case the $\vec{x}^{(2)}$ "component" of $\vec{x}$ approaches 0 faster than the $\vec{x}^{(1)}$ component and this is reflected in the diagram.

Note. If $\vec{x}' = A\vec{x}$ where A is Hermitian, then the eigenvalues of A are real and the eigenvalues are linearly independent. Therefore the general solution to the DE is

$$\vec{x} - c_a\vec{\xi}^{(1)}e^{R_1 t} + c_2\vec{\xi}^{(2)}e^{R_2 t} + \cdots + c_n\vec{\xi}^{(n)}e^{R_n t}$$

where $R_1, R_2, \ldots, R_n$ are the eigenvalues and $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \ldots, \vec{\xi}^{(n)}$ the corresponding eigenvectors.

Example. Page 358 Number 26. Consider the equation

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants with $a \neq 0$. Transform this into a system of first order equations and find the system $\vec{x}' = A\vec{x}$. Show that the eigenvalues are the roots of the auxiliary equation.

Solution. Let

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases} \text{ so that } \begin{cases} x_1' = x_2 \\ x_2' = -(1/a)x_2 - (c/a)x_1. \end{cases}$$

With A as required,

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} -\lambda & 1 \\ -c/a & -b/a - \lambda \end{vmatrix} = -\lambda(-b/a - \lambda) + c$$

$$= b\lambda/a + \lambda^2 + c/a = (1/a)(a\lambda^2 + b\lambda + c).$$

So λ is an eigenvalue of A if and only if it satisfies the auxiliary equation for the given DE.

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