Section 7.6. Complex Eigenvalues

Note. In this section we consider the case $\vec{x}' = A\vec{x}$ where the eigenvalues of A are non-repeating, but not necessarily real. We will assume that A *is* real.

Theorem. If A is real and R_1 is an eigenvalue of A where $R_1 = \lambda + i\mu$ and $\vec{\xi}^{(1)}$ is the corresponding eigenvector then $R_2 = \lambda - i\mu$ is also an eigenvalue and is corresponding eigenvector is $\vec{\xi}^{(2)} = \overline{\vec{\xi}^{(1)}}$.

Note. Recall that Cauchy's formula states

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Note. If $R_1 = \lambda + i\mu$ is an eigenvalue of A and $\vec{\xi}^{(1)} = \vec{a} + i\vec{b}$ is the eigenvector then a solution to $\vec{x}' - A\vec{x}$ is

$$\vec{x}^{(1)} = \vec{\xi}^{(1)} e^{R_1 t} = (\vec{a} + i\vec{b})e^{(\lambda+i\mu)t} = (\vec{a} + i\vec{b})e^{\lambda t}(\cos \mu t + i \sin \mu t)$$

$$= e^{\lambda t}(\vec{a} \cos \mu t - \vec{b} \sin \mu t) + i e^{\lambda t}(\vec{a} \sin \mu t + \vec{b} \cos \mu t).$$

For $R_2 = \lambda - i\mu$ and $\vec{\xi}^{(2)} = \vec{a} - i\vec{b}$ we find another solution is

$$\vec{x}^{(2)} = e^{\lambda t}(\vec{a} \cos \mu t - \vec{b} \sin \mu t) - i e^{\lambda t}(\vec{a} \sin \mu t + \vec{b} \cos \mu t) = \overline{\vec{x}^{(1)}}.$$

We can show that the real and imaginary parts of $\vec{x}^{(1)}$ (and $\vec{x}^{(2)}$) are linearly independent. Therefore we have the following.

Theorem. If A is real, and $R_1 = \lambda + i\mu$ and $R_2 = \lambda - i\mu$ are eigenvalues with corresponding eigenvectors $\vec{\xi}^{(1)} = \vec{a} + i\vec{b}$ and $\vec{\xi}^{(2)} = \vec{a} - i\vec{b}$ then two linearly independent solutions to $\vec{x}' = A\vec{x}$ are

$$\vec{u}(t) = e^{\lambda t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t) \text{ and } \vec{v}(t) = e^{\lambda t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t).$$

Example. Page 364 Number 2. Consider $\vec{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$. Express the general solution in terms of real-valued functions. Also draw a direction field, sketch a few of the trajectories and describe the behavior of the solutions as $t \to -\infty$.

Solution. We need the eigenvalues. Consider

$$\det(A - R\mathcal{I}) = \begin{vmatrix} -1-R & -4 \\ 1 & -1-R \end{vmatrix} = (-1-R)^2 + 4 = R^2 + 2r + 5,$$

and we find from $R^2 + 2R + 5 = 0$ that the eigenvalues are $R = \dfrac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$. So we take $\lambda = -1$ and $\mu = 2$, $R_1 = \lambda + i\mu = -1 + 2i$, and $R_2 = \lambda - i\mu = -1 - 2i$. For eigenvalue $R_1 = -1 + 2i$ we find a corresponding eigenvector $\vec{x}^{(1)}$ by considering $(A - R_1\mathcal{I})\vec{x}^{(1)} = \vec{0}$:

$$\left[\begin{array}{cc|c} -1-(-1+2i) & -4 & 0 \\ 1 & -1-(-1+2i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2i & -4 & 0 \\ 1 & -2i & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\rightsquigarrow}$$

$$\left[\begin{array}{cc|c} 1 & -2i & 0 \\ -2i & -4 & 0 \end{array}\right] \overset{R_2 \to R_2 + 2iR_1}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so we need $\begin{array}{rcl} x_1 - 2ix_2 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcl} x_1 & = & 2ix_2 \\ x_2 & = & x_2 \end{array}$ or with $t = x_2$ as a free variable,

$$\begin{aligned} x_1 &= 2it \\ x_2 &= t. \end{aligned}$$ We can take (with $t = 1$)

$$\vec{\xi}^{(1)} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \vec{a} + i\vec{b}.$$

We can find $\vec{\xi}^{(2)}$ by conjugation of $\vec{\xi}^{(1)}$:

$$\vec{\xi}^{(2)} = \overline{\vec{\xi}^{(1)}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - i \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \vec{a} - i\vec{b}.$$

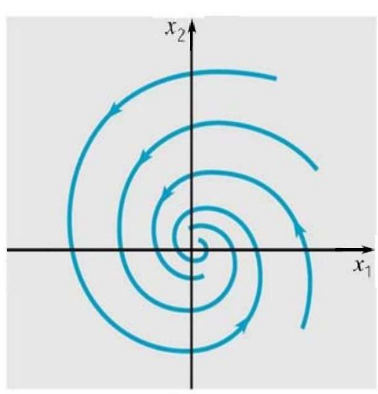
So the general solution is

$$\vec{x} = c_1 e^{\lambda} \left(\vec{a} \cos \mu t - \vec{b} \sin \mu t) \right) + c_2 e^{\lambda} \left((\vec{a} \sin \mu t + \vec{b} \cos \mu t) \right)$$

$$= c_1 e^{-t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t \right) + c_2 e^{-t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t \right)$$

$$= c_1 \begin{bmatrix} -2e^{-t} \sin 2t \\ e^{-t} \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} 2e^{-t} \cos 2t \\ e^{-t} \sin 2t \end{bmatrix}.$$

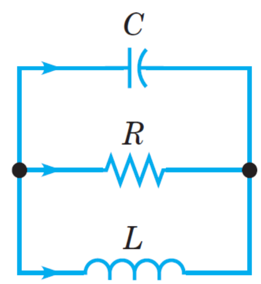
Since $\lambda < 0$, we see that all trajectories approach 0. Because of the trig functions, they spiral in counterclockwise. The following image is based on material in `http://www.math.nus.edu.sg/ matysh/ma3220/chap9.pdf`.



Example. Page 365 Number 14(a). The electric circuit shown below is described by the system of differential equations

$$\frac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. Show that the eigenvalues of the coefficient matrix are real and different if $L > 4R^2C$; show they are complex conjugates if $L < 4R^2C$.



Solution. For eigenvalues we consider:

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{vmatrix} = -\lambda\left(-\frac{1}{RC} - \lambda\right) + \frac{1}{CL} = \lambda^2 + \frac{1}{RC} + \frac{1}{CL}.$$

Setting this equal to 0 we find the eigenvalues

$$\lambda = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{CL}}}{2}.$$

So the eigenvalues are real and different if $\left(\frac{1}{RC}\right)^2 - \frac{4}{CL} > 0$ or $\frac{1}{R^2C^2} - \frac{4}{CL} > 0$ or $\frac{1}{R^2C^2} > \frac{4}{CL}$ or $R^2C^2 < \frac{CL}{4}$ or $4R^2C < L$, as claimed. The eigenvalues are complex conjugates if $\frac{1}{R^2C^2} - \frac{4}{CL} < 0$ or if $4R^2C > L$.