Section 7.7. Repeated Eigenvalues

Note. We again consider the system $\vec{x}' = A\vec{x}$. In this section we discuss the possibility that the eigenvalues of A are not distinct. We restrict ourselves to the special cases of A being 2×2 and 3×3.

Theorem. If $\vec{x}' = A\vec{x}$ where A is 2×2 and ρ is an eigenvalue of A of multiplicity 2, then one solution of the system is $\vec{x}^{(1)} = \vec{\xi} e^{\rho t}$ where $(A - \rho\mathcal{I})\vec{\xi} = \vec{0}$. If there is not a second (linearly independent) eigenvector satisfying $(A - \lambda_1\mathcal{I})\vec{c}_1 = \vec{0}$, then a second linearly independent solution is $\vec{x}^{(2)} = \vec{\xi} t e^{\rho t} + \vec{\eta} e^{\rho t}$ where $(A - \rho\mathcal{I})\vec{\eta} = \vec{\xi}$. In this case, $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ form a fundamental set of solutions of the DE.

Example. Page 371 Number 4. Consider $\vec{x}' = \begin{bmatrix} -3 & 5/2 \\ -5/2 & 2 \end{bmatrix} \vec{x}$. Find the general solution.

Solution. First, we find eigenvalues. Consider

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} -3-\lambda & 5/2 \\ -5/2 & 2-\lambda \end{vmatrix} = (-3-\lambda)(2-\lambda) + \frac{25}{4}$$

$$= \lambda^2 + \lambda - 6 + \frac{25}{4} = \lambda^2 + \lambda + \frac{1}{4} = \left(\lambda + \frac{1}{2}\right)^2.$$

So we have repeated eigenvalue $\lambda = \rho = -1/2$. We find that $\vec{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to this eigenvalue. So $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t/2}$ is a solution.

To find a second linearly independent solution, we consider the system of equations $(A - \rho\mathcal{I})\vec{\eta} = \vec{\xi}$ which leads to the augmented matrix

$$\begin{bmatrix} -5/2 & 5/2 & 1 \\ -5/2 & 5/2 & 1 \end{bmatrix} \underset{\sim}{\overset{R_2 \to R_2 - R_1}{}} \begin{bmatrix} -5/2 & 5/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{\overset{R_1 \to 2R_1}{}} \begin{bmatrix} -5 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

so we need $\begin{array}{rcrcl} -5\eta_1 & + & 5\eta_2 & = & 2 \\ & & 0 & = & 0 \end{array}$ or $\begin{array}{rcr} \eta_1 & = & -2/5 + \eta_2 \\ \eta_2 & = & \eta_2 \end{array}$ or with $t = \eta_2$ as a free variable, $\begin{array}{rcr} \eta_1 & = & -2/5 + t \\ \eta_2 & = & t. \end{array}$ With $t = 3/5$ we have $\eta = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is the desired vector. So another solutions is

$$\vec{x}^{(2)} = \vec{\xi} t e^{\rho t} + \vec{\eta} e^{\rho t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t/2} + \frac{1}{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t/2}.$$

So the general solution is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t/2} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t/2} + \frac{1}{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t/2} \right).$$

Theorem. If $\vec{x}' = A\vec{x}$ where A is 3×3 and ρ is an eigenvalue of A of multiplicity 3 then one solution of the system is $\vec{x}^{(1)} = \vec{\xi} e^{\rho t}$ where

$$(A - \rho\mathcal{I})\vec{\xi} = \vec{0}. \qquad (*)$$

If there is not a second (linearly independent) eigenvector satisfying $(*)$, then a second linearly independent solution is $\vec{x}^{(2)} = \vec{\xi} t e^{\rho t} + \vec{\eta} e^{\rho t}$ where $(A - \rho\mathcal{I})\vec{\eta} = \vec{\xi}$ and a third (linearly independent solution is

$$\vec{x}^{(3)} = \vec{\xi} \frac{t^2}{2!} e^{\rho t} + \vec{\eta} t e^{\rho t} + \vec{\zeta} e^{\rho t}$$

where $(A - \rho\mathcal{I})\vec{\zeta} = \vec{\eta}$.

Note. We will not consider the case of a 3×3 matrix with 2 linearly independent eigenvectors (see page 369 (24)–(27) and Page 372 Number 12).

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