Section 7.8. Fundamental Matrices

Note. We approach the system of equations $\vec{x}' = P(t)\vec{x}$ with a more direct use of matrices. We solve IVPs and address the exponentiation of matrices.

Definition. Suppose that $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}$ form a fundamental set of solutions for $\vec{x}' = P(t)\vec{x}$ on the interval $\alpha < t < \beta$. Then the matrix $\psi(t)$ whose columns are the vectors $\vec{x}^{(i)}$, for $i = 1, 2, \ldots, n$, is a *fundamental matrix* for the system of DEs.

Note. The general solution of $\vec{x}' = P(t)\vec{x}$ is $\vec{x} = \psi(t)\vec{c}$ where $\vec{c}$ is a vector of arbitrary constants. If the initial conditions $\vec{x}(t_0) = \vec{x}^0$ then we have $\psi(t_0)\vec{c} = \vec{x}^0$ and, since $\psi(t)$ is nonsingular (the columns are linearly independent), we have $\vec{c} = \psi^{-1}(t_o)\vec{x}^0$ and the general solution of the system is

$$\vec{x} = \psi(t)\psi^{-1}(t_0)\vec{x}^0.$$

Note. If we let $\varphi(t) = \psi(t)\psi^{-1}(t_0)$, then the general solution of the system is $\vec{x} = \varphi(t)\vec{x}^0$ and $\varphi(t_0) = \mathcal{I}$. In this case, any IVP can be solved simply by letting $\vec{x} = \varphi(t)\vec{x}^0$.

Example. Page 378 Number 5. Consider $\vec{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{x}$. Find the fundamental matrix $\varphi(t)$ where $\varphi(0) = \mathcal{I}$.

Solution. First, we find the eigenvalues and consider

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 2-R & -5 \\ 1 & -2-R \end{vmatrix} = (2-R)(-2-R) + 5 = -4 + R^2 + 5 = 1 + R^2,$$

and so the eigenvalues are $R = \pm i$. For the eigenvectors, consider $R_1 = i$ and the vector equation $(A - \lambda\mathcal{I})\vec{\xi} = \vec{0}$ which has the associated augmented matrix

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -(2+i) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and an eigenvector is

$$\vec{\xi}^{(1)} = \begin{bmatrix} 2+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda = 0$, $\mu = 1$, and the general solution to the DE is

$$\begin{aligned} \vec{x} &= c_1 e^{0t}(\vec{a}\cos 1t - \vec{b}\sin 1t) + c_2 e^{0t}(\vec{a}\sin 1t + \vec{b}\cos 1t) \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} (c_1 \cos t + c_2 \sin t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-c_1 \sin t + c_2 \cos t) \\ &= c_1 \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix}. \end{aligned}$$

So a fundamental matrix is

$$\psi(t) = \begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{bmatrix}$$

and

$$\psi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \psi^{-1}(0) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

Well, $\varphi(t) = \psi(t)\psi^{-1}(0)$, so

$$\varphi(t) = \begin{bmatrix} 2\sin t + \cos t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix}.$$

Notice that $\varphi(0) = \mathcal{I}$.

Example. Page 378 Number 5 (continued). Solve the IVP:

$$\begin{cases} \vec{x}\,' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{x} \\ \vec{x}^{\,0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{cases}$$

Solution. Well, $\vec{x} = \varphi(t)\vec{x}^{\,0}$, so

$$\vec{x} = \begin{bmatrix} -3\sin t + \cos t \\ \cos t - \sin t \end{bmatrix}.$$

Note. Other IVPs with the same DE, but different initial values, could be similarly solved.

Note. Consider $\vec{x}\,' = A\vec{x}$. If A is a diagonal matrix, then the system is very easy to solve (in this case, the variables are “uncoupled”). Suppose, though, that A is diagonalizable (recall that this is the case if A has a complete set of n linearly independent eigenvectors; in particular, this is true if A is Hermitian), say

$$D = T^{-1}AT = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix}$$

where the R_i are the eigenvalues of A. Consider the *change of variables* $\vec{x} = T\vec{y}$. Then $\vec{x}' = A\vec{x}$ becomes $T\vec{y}' = AT\vec{y}$ or $\vec{y}' = T^{-1}AT\vec{y} = D\vec{y}$. A fundamental matrix of this system is

$$Q(t) = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix}$$

and so a fundamental matrix of $\vec{x}' = A\vec{x}$ is $\psi = TQ$. This is the same solution as obtained in previous sections, but illustrates that the problem of solving a system of DEs is equivalent to diagonalizing a matrix.

Note. Recall that

$$e^{At} = \exp(At) = 1 + \sum_{n=1}^{\infty} \frac{(At)^n}{n!}.$$

So for a constant matrix A, we have the following.

Definition. If A is a constant matrix, define

$$\exp(At) = \mathcal{I} + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}.$$

Theorem. Each component of the above matrix sum converges as $n \to \infty$. This means that differentiation, integration, and multiplication can be done term-by-term.

Example. Notice that we have

$$\frac{d}{dt}[\exp(At)] = \frac{d}{dt}\left[\mathcal{I} + \sum_{n=1}^{\infty} \frac{A^n t^t}{n!}\right] = 0 + \sum_{n=1}^{\infty} \frac{A^n n t^{n-1}}{n!}$$

$$= A\left(\mathcal{I} + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}\right) = A\exp(At).$$

Note. The above example shows that a solution to $\vec{x}\,' = A\vec{x}$ is $\vec{x} = \exp(At)$. This means that the unique solution to the IVP

$$\begin{cases} \vec{x}\,' = A\vec{x} \\ \vec{x}\,'(0) = \vec{x}^{\,0} \end{cases}$$

is $\vec{x} = \exp(At)\vec{x}^{\,0}$.

Note. (This is not in the book.) Suppose A is diagonalizable, say $T^{-1}AT = D$ or $A = TDT^{-1}$. Then

$$\exp(At) = \exp(TDT^{-1}) = \mathcal{I} + \sum_{n=1}^{\infty} \frac{(TDT^{-1})^n t^n}{n!} = \mathcal{I} + T\left(\sum_{n=1}^{\infty} \frac{D^n t^n}{n!}\right)T^{-1}$$

$$= T\left(\mathcal{I} + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!}\right)T^{-1} = T\exp(Dt)T^{-1}.$$

So if A is diagonalizable, then the unique solution to

$$\begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}'(0) = \vec{x}^0 \end{cases}$$

is $\vec{x} = T\exp(Dt)T^{-1}\vec{x}^0$. Again, to find T, we need the eigenvalues and eigenvectors of A. We haven't saved any work here, only introduced some well motivated notation.

Example. Solve $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$ by this method (this is Page 356 Number 1).

Solution. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$ with eigenvectors $\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{\xi}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So

$$T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We find that $T^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$. So

$$\begin{aligned} \vec{x} &= T\exp(Dt)T^{-1}\vec{c} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\vec{c} \\ &= \frac{1}{3}\begin{bmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\vec{c} = \frac{1}{3}\begin{bmatrix} -e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ -2e^{-t} + 2e^{2t} & 4e^{-t} - e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{c_1}{3}(-e^{-t} + 4e^{2t}) + \frac{c_2}{3}(2e^{-t} - 2e^{2t}) \\ \frac{c_1}{3}(-2e^{-t} + 2e^{2t}) + \frac{c_2}{3}(4e^{-t} - e^{2t}) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \left(-\frac{c_1}{3} + \frac{2c_2}{3}\right) e^{-t} + \left(\frac{4c_1}{3} - \frac{2c_2}{3}\right) e^{2t} \\ \left(-\frac{2c_1}{3} + \frac{4c_2}{3}\right) e^{-t} + \left(\frac{2c_1}{3} - \frac{c_2}{3}\right) e^{2t} \end{bmatrix}$$

$$= k_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

where $k_1 = -c_1/3 + 2c_2/3$ and $k_2 = 2c_1/3 - c_2/3$.

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