Section 7.9. Nonhomogeneous Linear Systems

Note. We consider the nonhomogeneous system $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$ where $P(t)$ and $\vec{g}(t)$ are continuous for $\alpha < t < \beta$.

Definition. The *general solution* of $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$ is

$$c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)} + \vec{v}(t)$$

where $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)}$ is the general solution of the homogeneous equation $\vec{x}' = P(t)\vec{x}$ and $\vec{v}(t)$ is a *particular solution* of $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$.

Note. Now suppose $\psi(t)$ is a fundamental matrix for $\vec{x}' = P(t)\vec{x}$. We will use the method of *variation of parameters* to solve the nonhomogeneous DE in question. We will give a solution "symbolically" and the solution will not be expressed in a simple "closed form."

Note. Suppose there is a solution to the nonhomogeneous DE of the form $\vec{x} = \psi(t)\vec{u}(t)$. Then

$$\vec{x}' = \psi'(t)\vec{u}(t) + \psi(t)\vec{u}'(t)$$

and we get $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$ or

$$\psi'(t)\vec{u}(t) + \psi(t)\vec{u}'(t) = P(t)\psi(t)\vec{u}(t) + \vec{g}(t).$$

Since $\psi(t)$ is a fundamental matrix, then $\psi'(t) = P(t)\psi(t)$ and we get

$$\psi'(t)\vec{u}(t) + \psi(t)\vec{u}' = \psi'(t)\vec{u}(t) + \vec{g}(t)$$

or $\psi(t)\vec{u}'(t) = \vec{g}(t)$ or $\vec{u}'(t) = \psi^{-1}(t)\vec{g}(t)$ (remember, the columns of ψ are linearly independent so ψ^{-1} exists). Integrating,

$$\vec{u}(t) = \int \psi^{-1}(t)\vec{g}(t)\,dt + \vec{c}.$$

So we have

$$\vec{x} = \psi(t)\vec{u}(t) = \psi(t)\int \psi^{-1}(t)\vec{g}(t)\,dt + \psi(t)\vec{c}.$$

We summarize in a theorem.

Theorem. Consider $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$. Suppose $P(t)$ and $\vec{g}(t)$ are continuous for $\alpha < t < \beta$. Let $\psi(t)$ be a fundamental matrix of the homogeneous DE $\vec{x}' = P(t)\vec{x}$. Then the general solution fo the original nonhomogeneous DE is

$$\vec{x} = \psi(t)\vec{c} + \psi(t)\int \psi^{-1}(t)\vec{g}(t)\,dt$$

where $\vec{c}$ is a vector of arbitrary constants.

Note. If we consider the above DE with the added initial conditions $\vec{x}(t_0) = \vec{x}^0$ then we get the following theorem.

Theorem. Under the hypotheses of the previous theorem, the unique solution to the IVP

$$\begin{cases} \vec{x}' = P(t)\vec{x} + \vec{g}(t) \\ \vec{x}(t_0) = \vec{x}^0 \end{cases}$$

is

$$\vec{x} = \psi(t)\psi^{-1}(t_0)\vec{x}^0 + \psi(t)\int_{t_0}^{t} \psi^{-1}(s)\vec{g}(s)\,ds.$$

Example. Page 385 Number 8. Find the general solution of

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t.$$

Solution. The general solution of $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}$ is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

(this is Page 356 Number 3). So

$$\psi(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}.$$

From the requirement $\psi(t)\vec{u}'(t) = \vec{g}(t)$ we have

$$\begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

or

$$\left[\begin{array}{cc|c} e^t & e^{-t} & e^t \\ e^t & 3e^{-t} & -e^t \end{array}\right] \underset{\sim}{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} e^t & e^{-t} & e^t \\ 0 & 2e^{-t} & -2e^t \end{array}\right] \underset{\sim}{R_2 \to R_2/(2e^{-t})} \left[\begin{array}{cc|c} e^t & e^{-t} & e^t \\ 0 & 1 & -2e^{2t} \end{array}\right]$$

$$\underset{\sim}{R_1 \to R_1/e^t} \left[\begin{array}{cc|c} 1 & e^{-2t} & 1 \\ 0 & 1 & -2e^{2t} \end{array}\right] \underset{\sim}{R_1 \to R_1 - e^{-2t} R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -2e^{2t} \end{array}\right].$$

So we have $\vec{u}' = \begin{bmatrix} 2 \\ -e^{2t} \end{bmatrix}$. Then

$$\vec{u}(t) = \int \vec{u}'(t)\, dt = \int \begin{bmatrix} 2 \\ -e^{2t} \end{bmatrix} dt = \begin{bmatrix} 2t + c_1 \\ -\frac{1}{2}e^{2t} + c_2 \end{bmatrix}$$

and

$$\begin{aligned}
\vec{x} &= \psi(t)\vec{u}(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} 2t + c_1 \\ -\frac{1}{2}e^{2t} + c_2 \end{bmatrix} \\
&= \begin{bmatrix} 2te^t + c_1e^t - \frac{1}{2}e^t + c_2e^{-t} \\ 2te^t + c_1e^t - \frac{3}{2}e^t + 3c_2e^{-t} \end{bmatrix} \\
&= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} -\frac{1}{2} + 2t \\ -\frac{3}{2} + 2t \end{bmatrix} e^t.
\end{aligned}$$

Note. Again, consider $\vec{x}\,' = A\vec{x} + \vec{g}(t)$. Suppose A is diagonalizable, $D = T^{-1}AT$. Consider the change of variables $\vec{x} = T\vec{y}$. The DE becomes $T\vec{y} = AT\vec{y} + \vec{g}(t)$ or $\vec{y} = T^{-1}AT\vec{y} + T^{-1}\vec{g}(t)$ or $\vec{y} = D\vec{y} + \vec{h}(t)$ where $T^{-1}\vec{g}(t) = \vec{h}(t)$. Notice that this new system is uncoupled and the DEs can be solved one at a time by the methods of Introduction to Differential Equations.

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