Chapter 9. Nonlinear Differential Equations and Stability

Note. In this chapter we do not actually solve DEs but discuss, in a qualitative way, their behavior.

Section 9.1. The Phase Plane: Linear Systems

Note. In this section we consider $\vec{x}\,' = A\vec{x}$ where A is a 2×2 constant matrix.

Definition. A vector $\vec{x}$ satisfying $A\vec{x} = \vec{x}$ is called an *equilibrium solution* of the DE.

Note. At an equilibrium solution,

$$\frac{d\vec{x}}{dt} = \vec{x}\,' = A\vec{x} = \vec{0}.$$

Such points might also be called *critical points*. Notice that if $\det(A) \neq 0$ then the only equilibrium solution for the DE is $\vec{x} = \vec{0}$

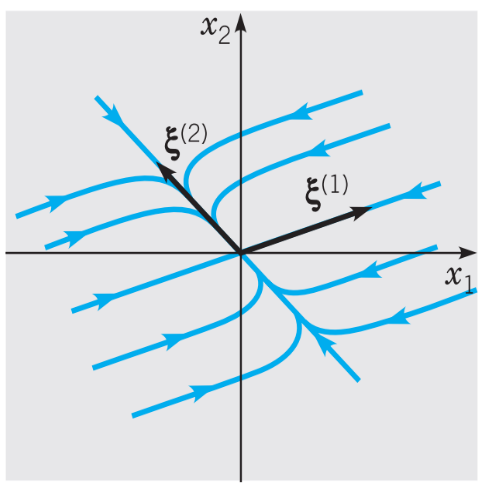
Definition. Suppose $\vec{x}$ is a solution to the DE. Then let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Recall that we can then represent any solution of the DE in the x_1, x_2-plane by a *trajectory* The x_1, x_2-plane is called the *phase plane* and the set of all trajectories is called the *phase portrait* of the DE.

Note. In this section, we relate the eigenvalues of A to the behavior of the trajectories.

Note. Consider $\vec{x}' = A\vec{x}$ where A is 2×2 and suppose the eigenvalues are real, distinct, and of the same sign. First, say $R_1 < R_2 < 0$. Then the general solution of the DE is

$$\vec{x} = c_1\vec{\xi}^{(1)}e^{R_1 t} + c_2\vec{\xi}^{(2)}e^{R_2 t}$$

and in the phase plane (this is Figure 9.1.1(a) in the 10 edition of DiPrima and Boyce):

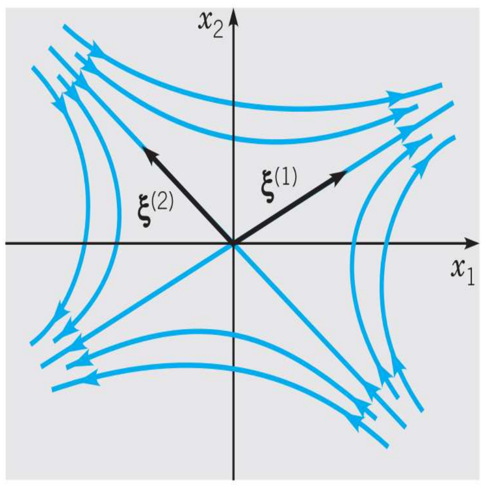


Notice that $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$ could point in any directions, but could not be parallel (or antiparallel). Since $R_1 < R_2 < 0$, any solution will satisfy $\vec{x} \to \vec{0}$ as $t \to \infty$. That is, solutions approach the equilibrium solution at $t \to \infty$. In this case, $\vec{0}$ is called a *node* or *nodal sink*. Second, say $0 < R_2 < R_1$. Then we have a similar case as above, except the directions of all trajectories are reversed. In this case, $\|\vec{x}\| \to \infty$ as $t \to \infty$. In this case, $\vec{0}$ is called a *node* or *nodal source*.

Note. Consider $\vec{x}' = A\vec{x}$ where A is 2×2 and suppose the eigenvalues are real, distinct, and of opposite signs. Suppose $R_1 > 0$ and $R_2 < 0$. Then the general solution is

$$\vec{x} = c_1 \vec{\xi}^{(1)} e^{R_1 t} + c_2 \vec{\xi}^{(2)} e^{R_2 t}$$

and in the phase plane (this is Figure 9.1.2(a) in the 10 edition of DiPrima and Boyce):

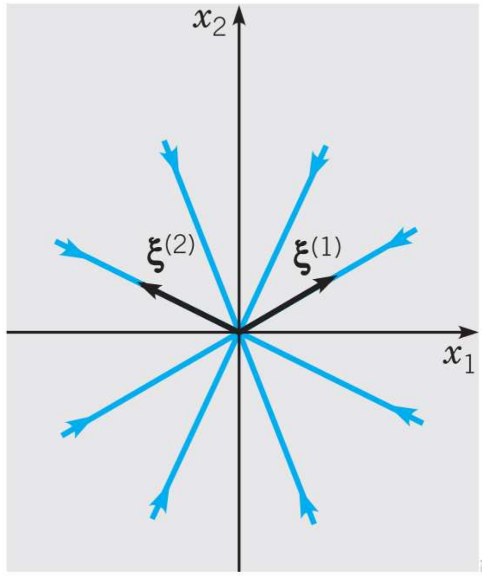


Notice that if $c_1 = 0$, then $\vec{x} \to \vec{0}$ as $t \to \infty$. If $c_1 \neq 0$ then $\|\vec{x}\| \to \infty$ as $t \to \infty$. In this case, the equilibrium $\vec{0}$ is called a *saddle point*.

Note. Consider $\vec{x}' = A\vec{x}$ where A is 2×2 and suppose eigenvalue R is of *algebraic multiplicity* 2 and that there are two linearly independent eigenvectors associated with eigenvalue R (that is, R is an eigenvector of *geometric multiplicity* 2). Then the general solution is

$$\vec{x} = c_1 \vec{\xi}^{(1)} e^{Rt} + c_2 \vec{\xi}^{(2)} e^{Rt}$$

and if $R < 0$ in the phase plane (this is Figure 9.1.3(a) in the 10 edition of DiPrima and Boyce):

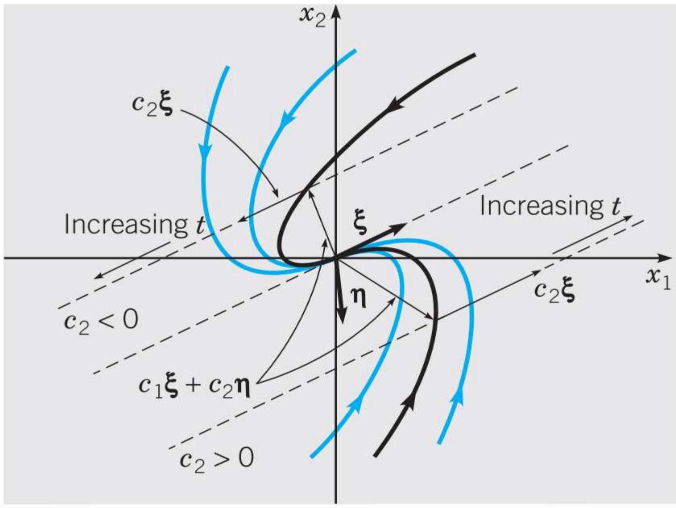


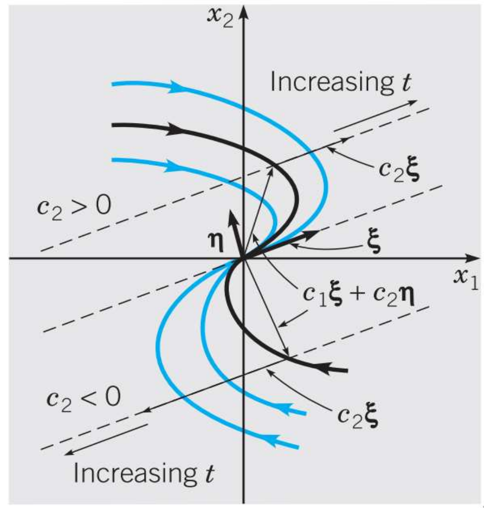
Notice that all trajectories are straight lines and $\vec{x} \to \vec{0}$ as $t \to \infty$. If $R > 0$ then we have a similar case as above, except the directions of all trajectories are reversed and in this case, $\|\vec{x}\| \to \infty$ as $t \to \infty$. With either $R < 0$ or $R > 0$, $\vec{0}$ is called a *proper node.*

Note. Consider $\vec{x}\,' = A\vec{x}$ where A is 2×2 and suppose eigenvalue R is of algebraic multiplicity 2 but that there is only one linearly independent eigenvector for R (that is, R is an eigenvector of geometric multiplicity 1). Then the general solution is

$$\vec{x} = c_1\vec{\xi}e^{Rt} + c_2\left(\vec{\xi}te^{Rt} + \vec{\eta}e^{rt}\right)$$

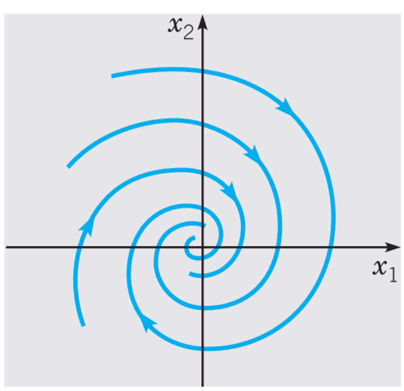
where $\vec{eta}$ is as discussed in Section 7.7, Repeated Eigenvalues." If $R < 0$ then in the phase plane (this is Figure 9.1.4(a) and (c) in the 10 edition of DiPrima and Boyce):

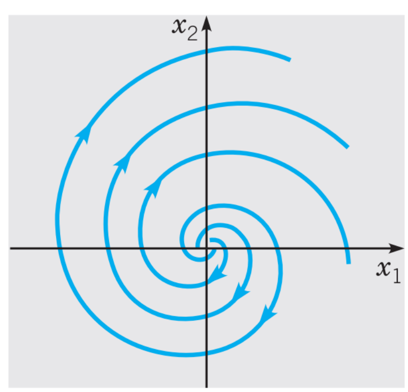




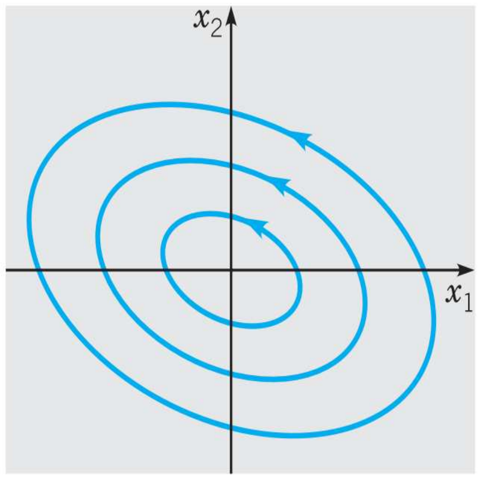
For t large, the dominant term of $\vec{x}$ is $c_2\vec{\xi}te^{Rt}$. See page 432 (see also Example 1 on pages 366–368). If $R < 0$, the arrows are simply reversed. In either case, $\vec{0}$ is called an *improper node* or *degenerate node*.

Note. Consider $\vec{x}' = A\vec{x}$ where A is 2×2 and suppose the eigenvalues of A are $\lambda \pm i\mu$ where $\lambda \neq 0$ and $\mu \neq 0$. As we saw in Chapter 7, trajectories in the phase plane of solutions are spirals. The critical point of such a system is called a *spiral point* (or sometimes *spiral sink/source*). In the phase plane (this is Figure 9.1.5(a) and (c) in the 10 edition of DiPrima and Boyce):





Note. Consider $\vec{x}' = A\vec{x}$ where A is 2×2 and suppose the eigenvalues of A are $\pm i\mu$. Again, from Chapter 7, since $\lambda = 0$ we get that the solutions are trajectories which are ellipses: In the phase plane (this is Figure 9.1.7(a) in the 10 edition of DiPrima and Boyce):



The ellipses are centered at $\vec{0}$ if the DE is homogeneous. In this case, $\vec{0}$ is called the *center*.

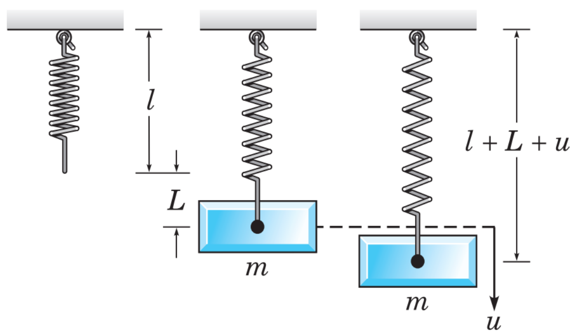
Note. In conclusion, for the 2×2 system $\vec{x}' = A\vec{x}$, the only critical point is $\vec{0}$. No two trajectories intersect and the only solution passing through $\vec{0}$ is the (unique) solution $\vec{0}$ (no other trajectories pass through $\vec{0}$). We have seen solutions fall into the following categories:

Asymptotic Stability: All solutions remain bounded and do not approach $\vec{0}$ as $t \to \infty$.

Stability: All solutions remain bounded and do not approach $\vec{0}$ as $t \to \infty$.

Instability: Some trajectories approach infinity as $t \to \infty$.

Example. Page 437 Number 17. If a mass m is suspended on a spring with spring constant k and the system oscillates in a medium which resists motion with a force proportional to velocity (with constant of proportionality c), then the position of m is give by $mu'' + cu' + ku = 0$ where derivatives are with respect to time and displacement u is measured from the equilibrium position.



Write this second order equation as a system of two first order equations for $x = u$ and $y = du/dt$. Show that $x = 0$, $y = 0$ is a critical point and analyze the nature and stability of the cirtical point as a function of the parameters m, c, and k.

Solution. Substituting we have

$$\begin{array}{r} x = u \\ y = u' \end{array} \text{ which implies } \left\{ \begin{array}{r} y = x' \\ y' = -\frac{c}{m}y - \frac{k}{m}x \end{array} \right.$$

or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Notice, since the DE is homogeneous, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only critical point. For eigenvalues, we consider

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m}-\lambda \end{vmatrix} = \lambda\left(\frac{c}{m}+\lambda\right)+\frac{k}{m} = \lambda^2+\frac{c}{m}\lambda+\frac{k}{m}.$$

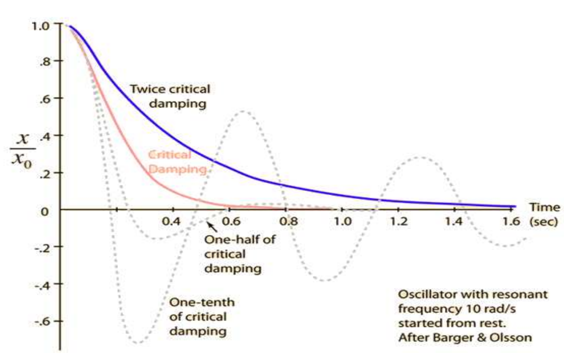
Setting this equal to 0 gives the eigenvalues $\lambda = \dfrac{-c \pm \sqrt{c^2-4mk}}{2m}$. We now analyze these eigenvalues.

1. If $c^2 - 4km > 0$ then the eigenvalues are real, distinct, and negative and the critical point is asymptotically stable. In this case, the motion is said to be *overdamped.*

2. If $c^2 - 4km < 0$ then the eigenvalues are complex with negative real parts and the critical point is an asymptotically stable spiral point. In this case the motion is said to be *underdamped.*

3. Of $c^2 - 4km = 0$ then the eigenvalues are real, equal, and negative. In either case, the critical point is a node. We find that there is only one linearly independent eigenvector for the given eigenvalue and so the node is improper. In this case the motion is said to be *cricitally damped.*

The graph of u as a function of time t for these different cases is:



This image is from http://hyperphysics.phy-astr.gsu.edu/hbase/oscda2.html (accessed 3/14/2019).

Revised: 3/14/2019