Section 9.2. Autonomous Systems and Stability

Note. We now consider a system of two DEs of the form

$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y), \end{cases}$$

where F and G are continuous and have continuous partial derivatives in some domain of the xy-plane (so if (x_0, y_0) is a point of the domain, the above DE along with the initial conditions $\begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$ has a unique solution of Theorem 7.1.1).

If we write the DE in vector form where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ then it becomes $\vec{x}\,' = \vec{f}(\vec{x})$.

Definition. A DE (or system of DEs) which does not explicitly involve the independent variable t is said to be *autonomous*. A DE which if not autonomous is said to be *nonautonomous*.

Example. The system $\vec{x}\,'$, where A is a constant matrix, is autonomous.

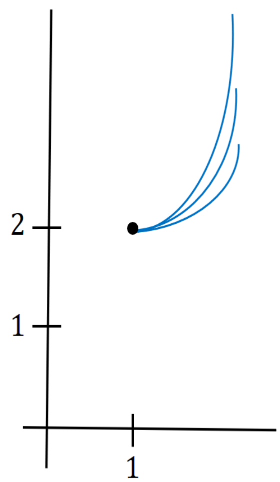
Note. On an autonomous system, all "particles" passing through a given point in the phase plane follow the same trajectory. This is what we saw in Section 9.1, but this does not hold for nonautonomous DEs. If

$$\begin{cases} \frac{dx}{dt} = \frac{x}{t}, \quad x(s) = 1 \\ \frac{dy}{dt} = -y, \quad y(s) = 2 \end{cases}$$

where x is some t value (initial time, say), then

$$\begin{cases} x = \frac{t}{x} \\ y = 2e^{t-2}. \end{cases}$$

Eliminating t gives $y = 2e^{s(x-1)}$. Notice that by taking different values for s we get several trajectories passing through $(x, y) = (1, 2)$:



Note. Notice that for

$$\begin{cases} \frac{dx}{dt} = F(x, y) \\ \frac{dy}{dt} = G(x, y), \end{cases}$$

we can write

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x, y)}{F(x, y)}$$

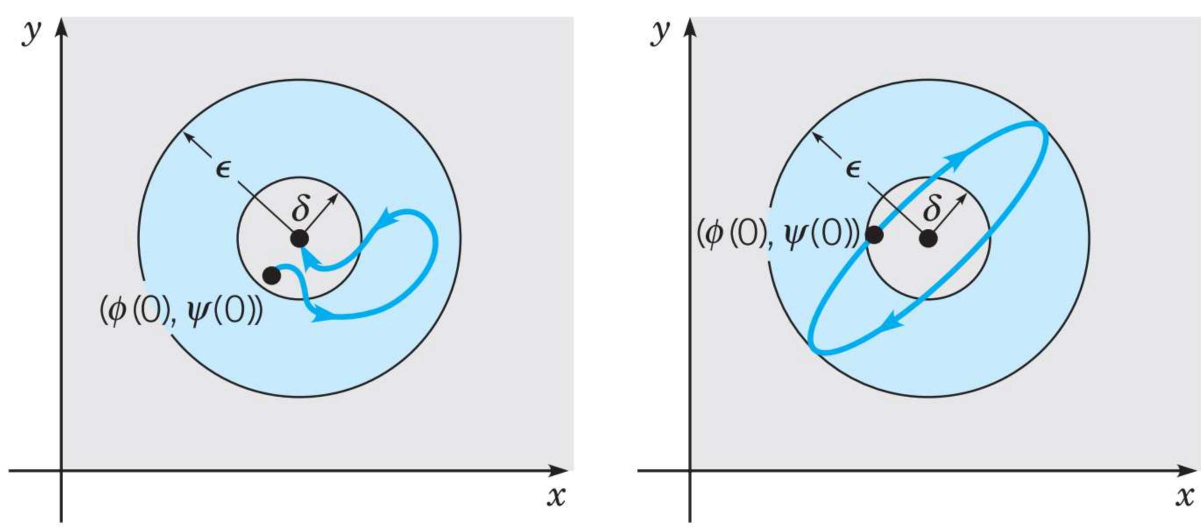
and we can (theoretically) derive a relationship between x and y which is independent of t.

Note. Physically, an autonomous system is one whose configuration is independent of time. The response of the system to initial conditions is independent of the time at which the conditions are imposed.

Note. We now put a rigorous definition on "stability." We consider an autonomous system $\vec{x}' = \vec{f}(\vec{x})$. If $\vec{f}(\vec{x}) = \vec{0}$, then $\vec{x}$ is called a *critical point*. At these points, we have an equilibrium.

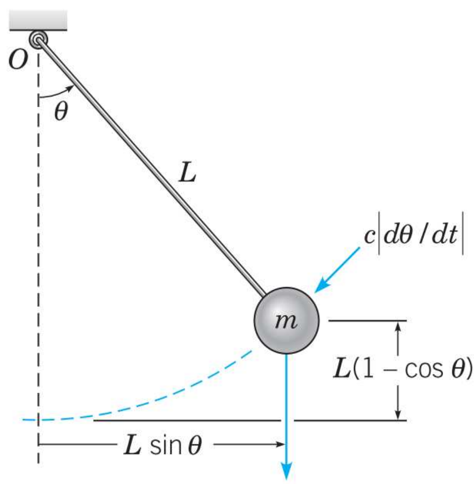
Definition. A critical point $\vec{x}^0$ of $\vec{x}' = \vec{f}(\vec{x})$ is said to be *stable* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that every solution $\vec{x} = \vec{\varphi}(t)$ which at $t = t_0$ satisfies $\|\vec{\varphi}(t_0) - \vec{x}'\| < \delta$ exists for all $t \geq t_0$ and satisfies $\|\vec{\varphi}(t) = \vec{x}'\| < \varepsilon$ for all $t \geq t_0$.

Note. The idea of stability is that if a solution "starts" near a critical point, then the solution stays near the critical point (but need not approach the critical point). The ε/δ relationship is illustrated in Figure 9.2.1 in the 10 edition of DiPrima and Boyce:



Definition. A critical point $\vec{x}^0$ is said to be *asymptotically stable* if it is stable and there exists $\delta_0 > 0$ such that if $\|\varphi(t_0) = \vec{x}^0\| < \delta_0$ then $\lim_{t\to\infty} \varphi(t) = \vec{x}^0$.

Example. Suppose a mass m is suspended from a rod of length ℓ and the system is oscillating (this is Figure 9.2.2 in the 10 edition of DiPrima and Boyce):



We saw in Introduction to Differential Equations that in the absence of a damping force,

$$m\ell^2 \frac{d^2\theta}{dt^2} = -mg\ell \sin\theta.$$

If we assume a damping force proportional to linear velocity (with constant of proportionality c) then this motion is described by

$$\text{angular momentum} = m\ell^2 \frac{d^2\theta}{dt^2} = -c\ell \frac{d\theta}{dt} = -mg\ell \sin\theta$$

or

$$\frac{d^2\theta}{dt^2} + \frac{c}{m\ell}\frac{d\theta}{dt} + \frac{g}{\ell}\sin\theta = 0.$$

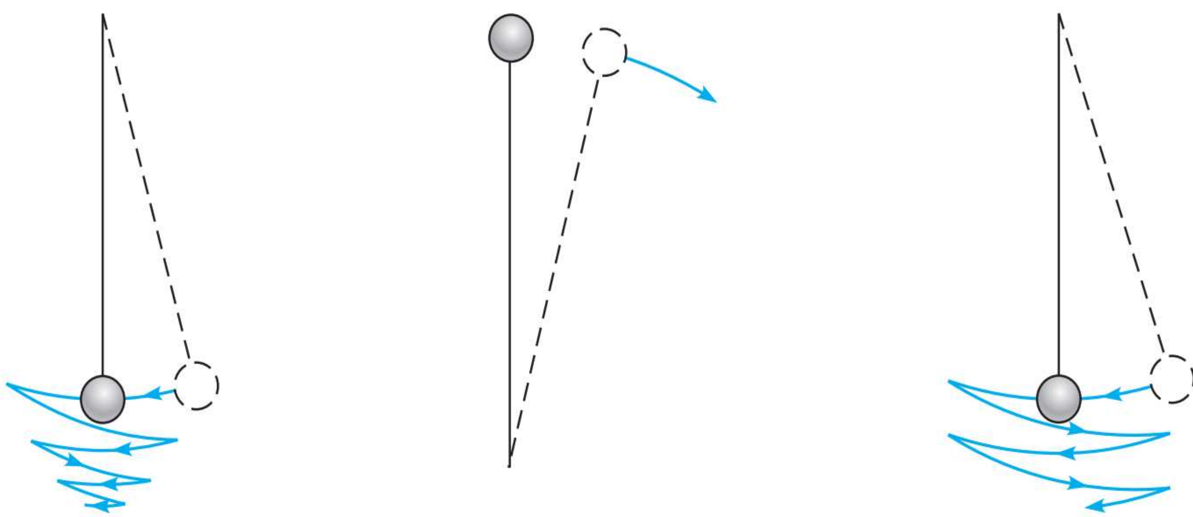
Notice that this DE is nonlinear because of the $\sin\theta$ term. Let $\begin{cases} x = \theta \\ y = \frac{d\theta}{dt}. \end{cases}$ Then we get

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{g}{\ell}\sin x - \frac{c}{m\ell}y. \end{cases}$$

Notice this nonlinear system is autonomous. For critical points, we have

$$\begin{cases} \frac{dx}{dt} = y = 0 \\ \frac{dy}{dt} = -\frac{g}{\ell}\sin x - \frac{c}{m\ell}y = 0 \end{cases}$$

or $y = 0$ and $x = n\pi$ where $n \in \mathbb{Z}$. So there are two states of the system which represent equilibria. One where the mass is at the lowest point and the other where the mass is at its highest point. We have:



Example. Page 446 Number 12. Show that the equation $\frac{g}{\ell}(1 - \cos x) + \frac{y^2}{2} = c$ describes trajectories of the DE

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0.$$

Solution. Differentiating with respect to x we get

$$\frac{d}{dx}\left[\frac{g}{\ell}(1 - \cos x) + \frac{y^2}{2}\right] = \frac{d}{dx}[c]$$

or

$$\frac{g}{\ell}\sin x + y\frac{dy}{dx} = 0. \qquad (*)$$

With $x = \theta$ and $y = d\theta/dt$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dy/dt}{y} = \frac{d^2\theta/dt^2}{y}$$

or $y\dfrac{dy}{dx} = \dfrac{d^2\theta}{dt^2}$. So by $(*)$ we have

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

and the original equation describes trajectories of this DE.

Example. Page 446 Number 16. Suppose (x_0, y_0) is a critical point of

$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y). \end{cases}$$

Prove that a trajectory of the DE cannot reach (x_0, y_0) from a noncritical point in finite time.

Proof. Suppose, to the contrary, $x = \varphi(t)$ and $y = \psi(t)$ is a nonconstant solution, $\varphi(a) = x_0$ and $\psi(a) = y_0$. By Page 446 Number 15 there is a unique trajectory passing through a given point. However, the constant solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is also a "trajectory" containing the point (x_0, y_0), a contradiction. So solutions of the DE cannot reach a critical point from a noncritical point in finite time. ∎

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