Section 9.3. Almost Linear Systems

Note. We now "linearize" almost linear systems of differential equations and use the linearization to analyze the stability of critical points of the almost linear system.

Theorem 9.3.1. The critical point $\vec{x} = \vec{0}$ of the linear system $\vec{x}\,' = A\vec{x}$ is:

1. asymptotically stable if the eigenvalues R_1 and R_2 of A are real and negative, or have negative real part,

2. stable, but not asymptotically stable, if R_1 and R_2 are purely imaginary,

3. unstable if R_1 and R_2 are real and at least one is positive, or if at least one has positive real part (of course with A real and R_1 complex then $R_1 = \overline{R_2}$).

Note. If the entries of A are slightly perturbed, then a stable system can be changed into an unstable one. If $R_1 = \overline{R_2} = i\mu$, then the critical point is a center. If the entries of A are slightly perturbed, then R_1 and R_2 will be slightly perturbed. So the critical point could become a spiral point and the system may be stable or unstable (depending on whether the real part of R_1 and R_2 is positive or negative).

Note. If $R_1 = R_2$ then a perturbation could convert the critcal point (a node) into either a node (with the new eigenvalues still real, an unlikely event) or a spiral point. In either case, with *small* perturbations, the stability is not changed, although the trajectories may be significantly different. In the remaining cases, small perturbations do not affect stability.

Note. We no consider nonlinear systems which behave like linear systems near their critical points. For the autonomous system $\vec{x}' = \vec{f}(\vec{x})$, if $\vec{x}^0$ is a critical point, then by making the substitution $\vec{u} = \vec{x} - \vec{x}^0$, we can convert this system into one with $\vec{0}$ as a critical point.

Definition. Suppose $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$ where A is a constant matrix, $\det(A) \neq 0$ and $\vec{x} = \vec{0}$ is an *isolated* critical point of the DE. Suppose the components of $\vec{g}$ have continuous first partial derivatives and

$$\lim_{\vec{x}\to\vec{0}} \frac{\|\vec{g}(\vec{x})\|}{\|\vec{x}\|} = 0.$$

Then the DE is said to be *almost linear* in a neighborhood of $\vec{0}$.

Note. The condition $\displaystyle\lim_{\vec{x}\to\vec{0}} \frac{\|\vec{g}(\vec{x})\|}{\|\vec{x}\|} = 0$ is equivalent to

$$\begin{cases} \lim_{r\to 0} \frac{g_1(x,y)}{r} = 0, \text{ and} \\ \lim_{r\to 0} \frac{g_2(x,y)}{r} = 0 \end{cases}$$

where

$$\vec{g}(\vec{x}) = \begin{bmatrix} g_1(\vec{x}) \\ g_2(\vec{x}) \end{bmatrix} \text{ and } r = \|\vec{x}\| = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{x^2 + y^2}.$$

Note. Stability of the critical point $\vec{0}$ for an almost linear system is given in terms of the eigenvalues of A in Theorem 9.3.2 and Table 9.3.1.

Example. Consider

$$\begin{cases} \frac{dx}{dt} = y + x(1 - x^2 - y^2) \\ \frac{dy}{dt} = -x + y(1 - x^2 - y^2). \end{cases}$$

Verify that $\vec{0}$ is a critical point, the system is almost linear, and discuss the stability of $\vec{0}$.

Solution. Clearly $\vec{0}$ is a critical point. Also,

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} -x^3 - xy^2 \\ -x^2y - y^3 \end{bmatrix}.$$

Letting $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\lim_{r\to 0} \frac{g_1(x,y)}{r} = \lim_{r\to 0} \frac{-r^3(\cos^3\theta - \cos\theta\sin\theta)}{r} = 0$$

and

$$\lim_{r\to 0} \frac{g_2(x,y)}{r} = \lim_{r\to 0} \frac{r^3(-\cos^2\theta\sin\theta - \sin^3\theta)}{r} = 0.$$

So the system is almost linear near $\vec{0}$ (notice that partials are continuous). Now

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$$

and so $\lambda = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$ are the eigenvalues. So $\vec{0}$ is a spiral point and the system is unstable.

Example. Recall that the DE for a damped pendulum is

$$m\ell^2 \frac{d^2\theta}{dt^2} = -c\ell\frac{d\theta}{dt} - mg\ell\sin\theta,$$

which can be written

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{g}{\ell}\sin x - \frac{c}{m\ell}y \end{cases}$$

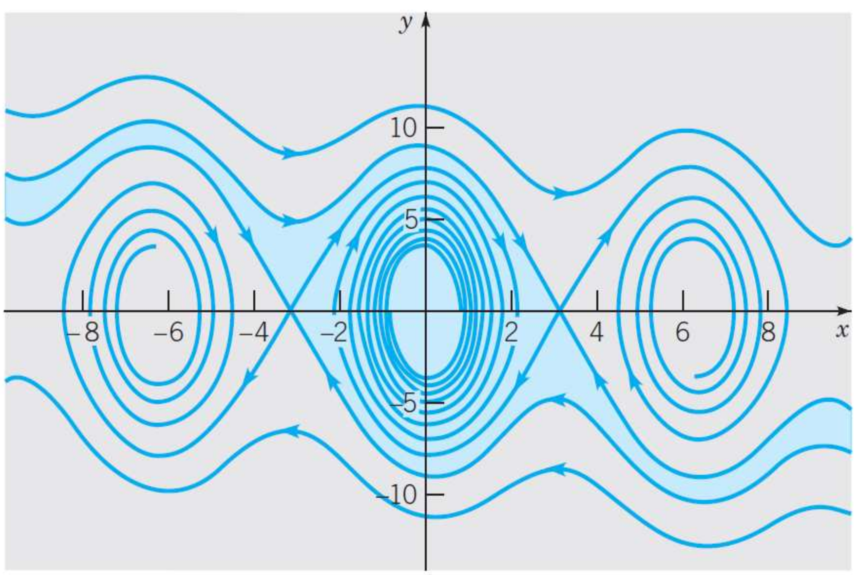
where $x = \theta$ and $y = d\theta/dt$. We can also write

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{g}{\ell}x - \frac{c}{m\ell}y - \frac{g}{\ell}(\sin x - x) \end{cases}$$

or

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{c}{m\ell} \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ -\frac{g}{\ell}(\sin x - x) \end{bmatrix}.$$

It can be shown that $\lim\limits_{r\to 0}\dfrac{\sin x - x}{r} = 0$ (see Page 450 Example 2). So this system is almost linear near $\vec{0}$. As in the previous section, we find that this system has asymptotically stable critical points at $2n\pi$ where $n \in \mathbb{Z}$, and unstable critical points at $(2n+1)\pi$ where $n \in \mathbb{Z}$. In the phase plane (this is Figure 9.3.5 in the 10 edition of DiPrima and Boyce):



Note/Definition. Notice that in certain regions of the phase plane, solutions are attracted to stable equilibria. These regions are called the *basin of attraction* (or region of asymptotic stability) for the critical point. A boundary between these regions is called a *separatrix*. If every trajectory approaches a critical point, as with the system $\vec{x}' = A\vec{x}$ for the critical point $\vec{0}$, then the critical point is said to be *globally asymptotically stable.*

Revised: 3/16/2019