Section 9.4. Competing Species

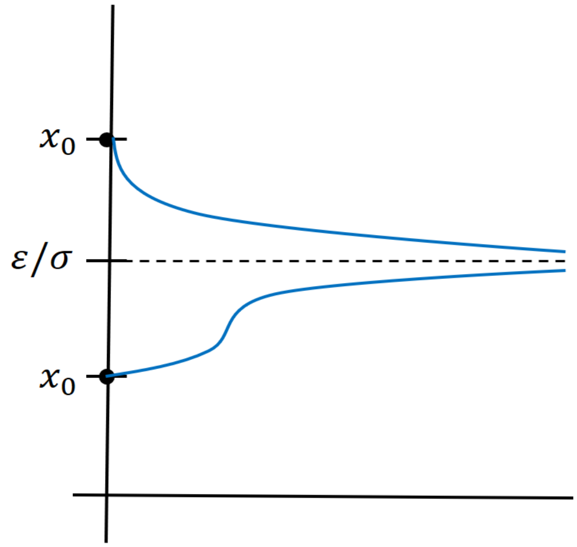
Note. You saw in Calculus 2 the idea of exponential growth and decay; see my online notes for "Exponential Change and Separable Differential Equations" at `http://faculty.etsu.edu/gardnerr/1920/12/c7s2.pdf`. We can modify the idea of a population that grows exponentially by introducing a carrying capacity on the environment. This leads to the logistic equation (also see my notes on application of first-order DES from Calculus 2: `http://faculty.etsu.edu/gardnerr/1920/12/c9s3.pdf`). In this section, we briefly review these ideas and then introduce a model concerning two species in an environment which compete with one another for resources. This is modeled by two DEs in two variables.

Note/Definition. If x is the size of a population at time t and if the rate of growth of x is proportional to x then $dx/dt = \varepsilon x$ and $x = x_0 e^{\varepsilon t}$. This is called *exponential growth* (or "exponential decay" if $\varepsilon < 0$).

Note/Definition. If we put a maximum on the size of x (usually called a *carrying capacity*) then we have $dx/dt = x(\varepsilon - \sigma x)$ and

$$x = \frac{x_0(\varepsilon/\sigma)}{x_0 + ((\varepsilon/\sigma) - x_0)e^{-\varepsilon t}}.$$

This is called *logistic growth*.



Note. We now consider two species which interact (compete) and are governed by logistic growth. We have

$$\begin{cases} \frac{dx}{dt} = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ \frac{dy}{dt} = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{cases}$$

In order to analyze this system, we need the following theorem.

Theorem. Consider $\vec{x}\,' = \vec{f}(\vec{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$. Suppose (x_0, y_0) is a critical point. Then (by Taylor's Theorem for expansions of F and G about the point (x_0, y_0), see page 450)

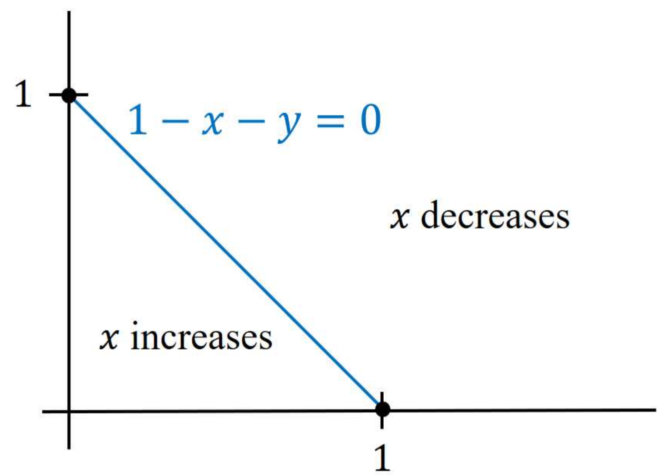
$$\vec{x}\,' = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{bmatrix}$$

for some η_1 and η_2. Here $F_x = \partial F/\partial x$ and $F_y = \partial F/\partial y$, and similarly for G.

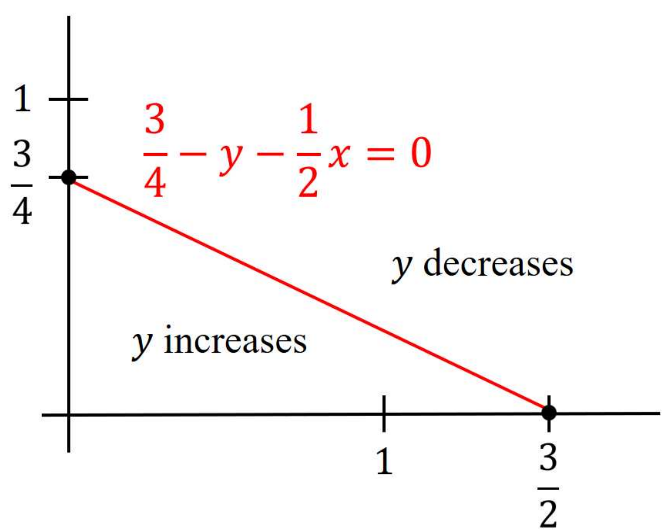
Example. Page 460 Example 1. Describe the qualitative behavior of solutions of the system

$$\begin{cases} \frac{dx}{dt} = x(1 - x - y) \\ \frac{dy}{dt} = y(\frac{3}{4} - y - \frac{1}{2}x). \end{cases}$$

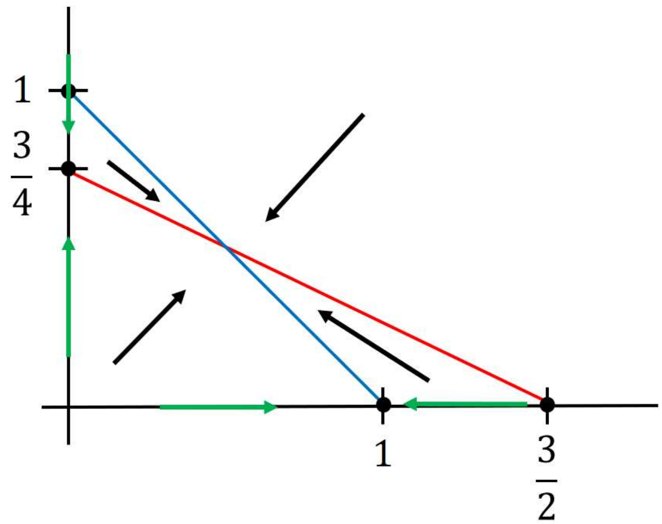
Solution. Well, we find that the critical points are $(0,0)$, $(0,3/4)$, $(1,0)$, and $(1/2,1/2)$. Notice that along the line $1 - x - y = 0$, we have $dx/dt = 0$:



This line is called the x *nullcline*. Similarly, the y nullcline is the line $\frac{3}{4} - y - \frac{1}{2}x = 0$:



Graphing the x and y nullclines together:



We have:

1. For critical point $(0,0)$, we have

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \vec{x} + \begin{bmatrix} -x^2 - xy \\ -\frac{1}{2}xy - y^2 \end{bmatrix}$$

and the eigenvalues of the matrix A are 1 and 3/4. So $(0,0)$ is an unstable node.

2. For critical point $(1,0)$ we have by the theorem above that

$$\vec{x}' = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \vec{\eta}.$$

Note that the system is almost linear near $(1,0)$ also. We have the eigenvalues of A of -1 and $1/4$, so $(1,0)$ is a saddle point.

3. For critical point $(0,3/4)$, we have

$$\vec{x}' = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \vec{\eta}$$

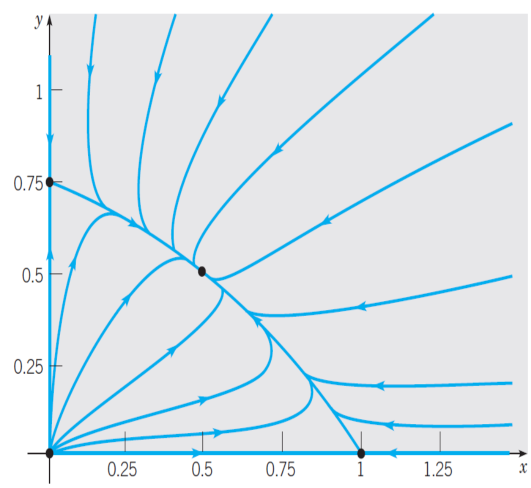
and the eigenvalues are $1/4$ and $-3/4$, so $(0, 3/4)$ is also a saddle point.

4. For critical point $(1/2, 1/2)$, we have

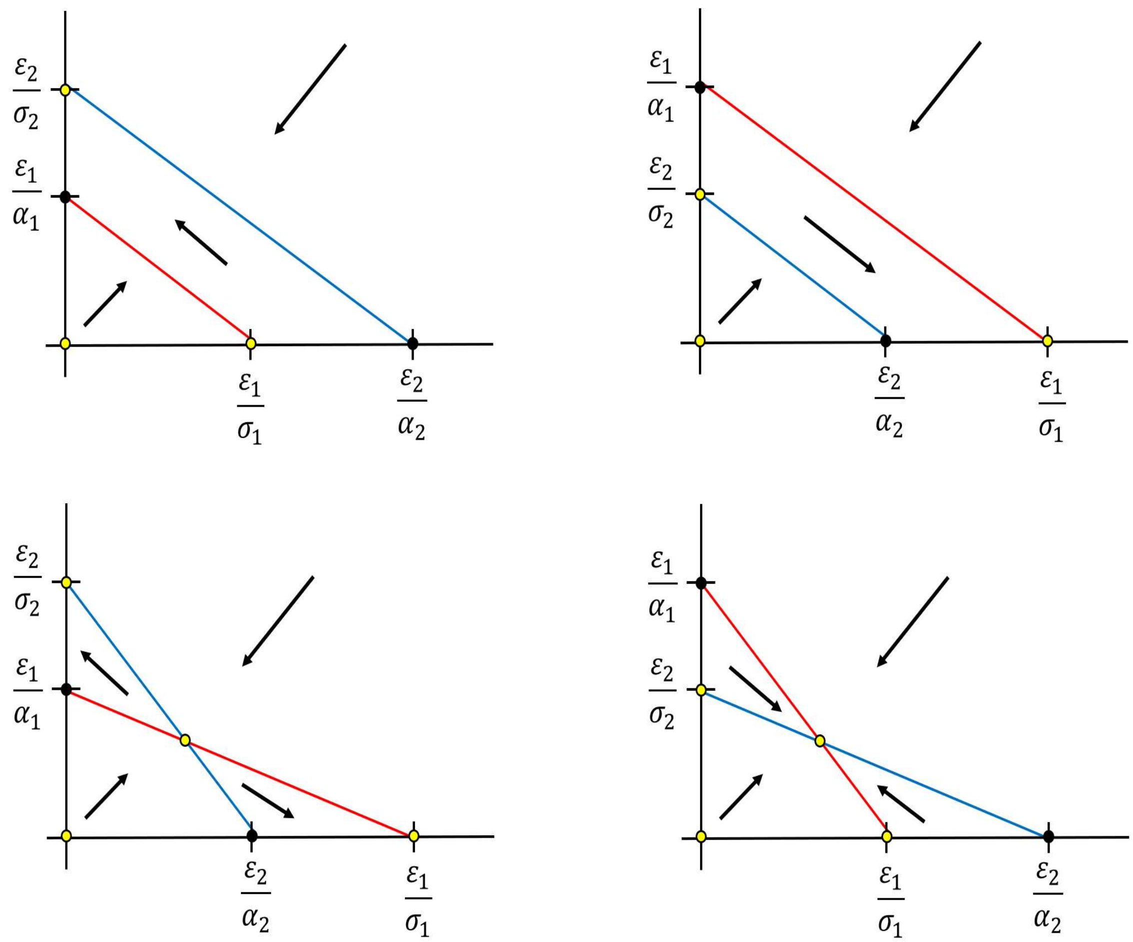
$$\vec{x}\,' = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \vec{\eta}$$

and the eigenvalues are $(-2 \pm \sqrt{2})/4$ and so $(1/2, 1/2)$ is an asymptotically stable node.

Combining this information, we have (this is Figure 9.4.2 in the 10 edition of DiPrima and Boyce):



Note. The previous example suggests that we have 4 possibilities according to the values of $\varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2, \alpha_1, \alpha_2$ (equilibria are represented with yellow circles):



Note. An analysis of the eigenvalues shows that:

1. if $\sigma_1\sigma_2 > \alpha_1\alpha_2$ then the competition is weak and the species can coexist,

2. if $\sigma_1\sigma_2 < \alpha_1\alpha_2$ then the competition is strong and the species cannot coexist.

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