Section 9.5. Predator-Prey Equations

Note. We now assume two interacting species, x of prey and y of predators. We assume:

1. In the absence of predators, the prey grow exponentially,

2. in the absence of prey, the predators die off exponentially, and

3. the number of encounters between predators and prey is proportional to xy. The growth rate of the predators is enhanced by encounters and the growth rate of the prey is hindered by encounters.

In this section, we describe a system of two DEs in two dependent variables based on these ideas. We find the critical points of the resulting system and classify their stability.

Definition. The model described above leads us to the *Lotka-Volterra Predator Prey Equations*:

$$\begin{cases} \frac{dx}{dt} = ax - \alpha xy = x(a - \alpha y) \\ \frac{dy}{dt} = -cy + \gamma xy = y(-c + \gamma x). \end{cases}$$

Note. Notice that the critical points of the Lotka-Volterra Equations are $(0,0)$ and $(c/\gamma, a/\alpha)$. The system can be written

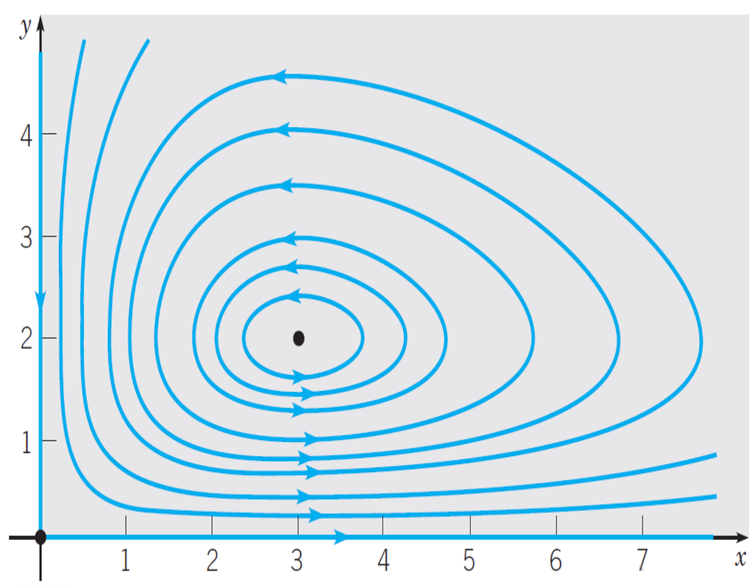
$$\vec{x}\,' = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \vec{x} + \begin{bmatrix} -\alpha xy \\ \gamma xy \end{bmatrix}.$$

Notice that the system is almost linear near $(0,0)$. The eigenvalues of $\begin{bmatrix} 1 & 0 \\ 0 & -c \end{bmatrix}$ are a and $-c$. So $(0,0)$ is a saddle point and so it is unstable. Making the substitutions $\begin{cases} u = x - c/\gamma \\ v = y - a/\alpha \end{cases}$ we have

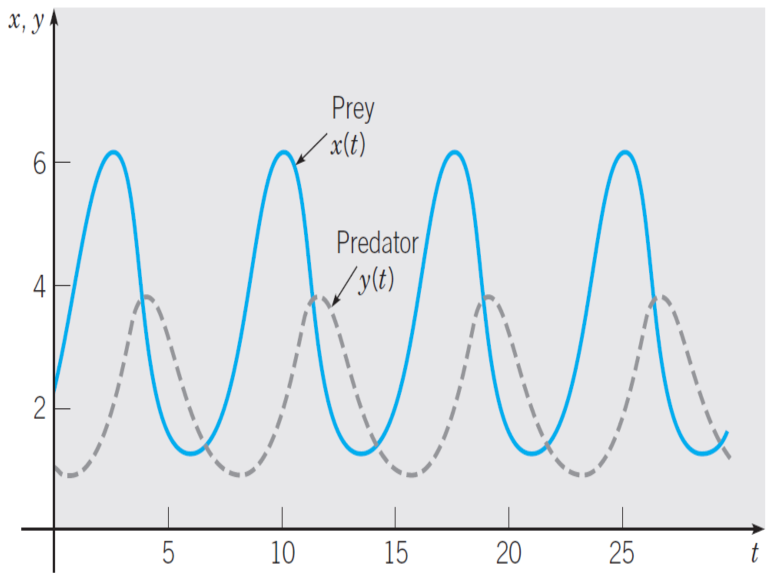
$$\begin{bmatrix} du/dt \\ dv/dt \end{bmatrix} = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} a\left(u + \frac{c}{\gamma}\right) - \alpha\left(u + \frac{c}{\gamma}\right)\left(v + \frac{a}{\alpha}\right) \\ -c\left(v + \frac{a}{\alpha}\right) + \gamma\left(u + \frac{c}{\gamma}\right)\left(v + \frac{a}{\alpha}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha uv - \frac{\alpha c}{\gamma} v \\ \gamma uv + \frac{\gamma a}{\alpha} u \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\alpha c}{\gamma} \\ \frac{\gamma a}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -\alpha uv \\ \gamma uv \end{bmatrix}.$$

Notice that this system is almost linear at $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, in which case $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c/\gamma \\ a/\alpha \end{bmatrix}$. The eigenvalues of $\begin{bmatrix} 0 & -\frac{\alpha c}{\gamma} \\ \frac{\gamma a}{\alpha} & 0 \end{bmatrix}$ are $\pm i\sqrt{ac}$. So $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c/\gamma \\ a/\alpha \end{bmatrix}$ is a center and therefore a stable critical point. So in the xy-plane (this is Figure 9.5.2 in the 10 edition of DiPrima and Boyce):



In terms of time (this is Figure 9.5.3 in the 10 edition of DiPrima and Boyce):



Notice that the prey react to increases and decreases of the prey, but with a delay. The prey, in return react to increases and decreases of predators, but with a delay.

Note. Near the center, we find that trajectories in the xy-plan are ellipses. If we consider

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(-c+\gamma x)}{x(a-\alpha y)},$$

then we get $a \ln y - \alpha y + c \ln c - \gamma x = c$ where c is a constant. The graphs of these trajectories are closed curves around the critical point $(c/\gamma, a/\alpha)$.

Revised: 3/17/2019