Section 9.6. Liapunov's Second Method

Note. In a physical, dynamical system, such as the pendulum or mass on a spring, the total energy is constant (assuming a conservative system, of course). Also, an equilibrium is stable if the potential energy is at a local minimum. With this as inspiration, we look for a function V which will behave somewhat like total energy. We will call such a function a Liapunov function.

Definition. Let V be defined on some domain D (i.e., an open connected set) in $\mathbb{R}^2$ containing $(0,0)$. V is *positive definite* on D if $V(0,0) = 0$ and $V(x,y) > 0$ for all other $(x,y) \in D$. If $V(0,0) = 0$ and $V(x,y) \geq 0$ for all $(x,y) \in D$ then V is *positive semidefinite*. Negative definite and semidefinite are similarly defined (with $<$ and $\leq$).

Definition. Consider the autonomous system

$$(*) \qquad \begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$$

and a function $V(x,y)$. Define

$$\dot{V} = V_x(x,y)F(x,y) + V_y(x,y)G(x,y).$$

$\dot{V}$ is the *derivative of* V *with respect to the system* $(*)$.

Note. $\dot{V}$ is the rate of change of V along a trajectory of $(*)$ that passes through the point (x,y).

Theorem 9.6.1. Suppose $(*)$ has an isolated critical point at $(0,0)$. If there exists a V that is continuous and has continuous first partial derivatives, is positive definite, and $\dot{V}$ is negative definite on some domain D in $\mathbb{R}^2$ containing $(0,0)$ then $(0,0)$ is an asymptotically stable critical point. If $\dot{V}$ is negative semidefinite then $(0,0)$ is a stable critical point.

Theorem 9.6.2. Suppose $(*)$ has an isolated critical point at $(0,0)$. let V be continuous with continuous first partial derivatives. Suppose $V(0,0) = 0$ and that is every neighborhood of $(0,0)$ there is a point at which V is positive (negative). Then if there is a domain D containing $(0,0)$ such that $\dot{V}$ is positive definite (negative definite) on D, then $(0,0)$ is an unstable critical point.

Note. The idea of Theorem 9.6.1: Think of V as total energy. If V is positive definite and $\dot{V}$ is negative definite, then V has a local minimum at $(0,0)$. With $\dot{V}$ negative definite, energy is strictly decreasing along trajectories approach $(0,0)$. With $\dot{V}$ negative semidefinite, we only know that the trajectories do not go away from $(0,0)$.

Note. The idea of Theorem 9.6.2: If $V(0,0) = 0$ and $\dot{V}$ is positive definite, then energy increases along trajectories and so trajectories go away from $(0,0)$ since V is positive "away from" $(0,0)$. Similarly with $\dot{V}$ negative definite (energy decreases along trajectories).

Definition. A function V satisfying the conditions of Theorems 9.6.1 and 9.6.2 is called a *Liapunov function.*

Note. We can use Liapunov functions to find basins of attraction for asymptotically stable critical points, as in the following theorem.

Theorem 9.6.3. Let $(0,0)$ be an isolated critical point of $(*)$. Let V be continuous and have continuous first partial derivatives. If there is abounded domain D_K containing $(0,0)$ where $V(x,y) < K$, V is positive definite, and $\dot{V}$ is negative definite, then an energy solution that starts in D_K approaches $(0,0)$ as $t \to \infty$. That is, D_K is in the basin of attraction of $(0,0)$.

Note. Notice that we have nowhere said anything about the construction of Liapunov functions.

Example. Page 489 Number 1. Find a Liapunov function of the form $V(x,y) = ax^2 + cy^2$ and show

$$\begin{cases} \frac{dx}{dt} = -x^3 + xy^2 = F(x,y) \\ \frac{dy}{dt} = -2x^2y - y^3 = G(x,y) \end{cases}$$

has an asymptotically stable critical point at $(0,0)$.

Solution. Well, $(0,0)$ is certainly a critical point and $V(x,y)$ is positive definite in domains containing $(0,0)$ (excluding $(0,0)$) if $a > 0$ and $c > 0$. Now

$$\dot{V}(x,y) = V_x(x,y)F(x,y) + V_y(x,y)G(x,y) = -2ax^4 + (2a - 4c)x^2y^2 - 2cy^4.$$

So, $\dot{V}$ is negative definite if $2a < 4c$. So let $a = 1$ and $c = 2$. Then let $V(x,y) = x^2 + 2y^2$. By Theorem 9.6.1, this Liapunov function shows that $(0,0)$ is an asymptotically stable critical point.

Example. Page 490 Number 8. The Liénard equation is

$$\frac{d^2u}{dt^2} + c(u)\frac{du}{dt} + g(u) = 0,$$

$c(u) \geq 0$, where $g(0) = 0$, $g(u) > 0$ for $9 < u < k$ and $g(u) < 0$ for $-k < u < 0$. Show that $u = 0$, $du/dt = 0$ is a stable equilibrium.

Solution. Well, let $x = u$ and $y = du/dt$. Then the system becomes:

$$\begin{cases} \frac{dx}{dt} = y = F(x,y) \\ \frac{dy}{dt} = -c(x)y - g(x) = G(x,y). \end{cases}$$

The critical point is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Consider (see Page 490 Number 6):

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)\,ds, \ -k < x < k.$$

Notice $V(x,y)$ is positive definite for $-k < x < k$ and $-\infty < y < \infty$. Also

$$\dot{V}(x,y) = g(x)y + y(-c(x)y - g(x)) = -y^2c(x).$$

Notice $\dot{V}$ is negative semidefinite, so by Theorem 9.6.1, the critical point is stable.